Thursday, November 14 $**$ *Green's Theorem*

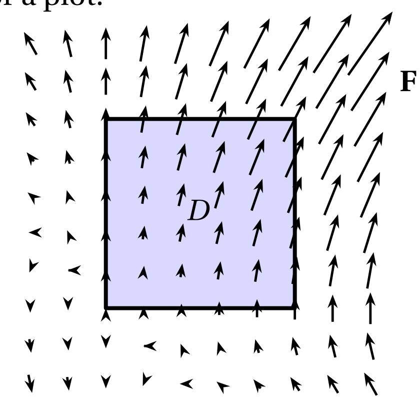
Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus: it relates the (integral of) a vector field $\mathbf{F}$ on the boundary of a region D to the integral of a suitable *derivative* of $\mathbf{F}$ over the whole of D.

1. Let D be the unit square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ and consider the vector field $\mathbf{F}(x,y) = \langle P(x,y),\ Q(x,y)\rangle = \langle xy,\ x+y\rangle$. See below right for a plot.

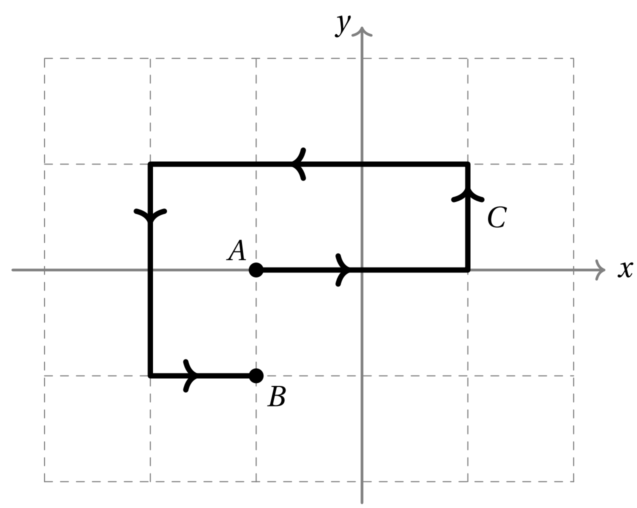
   (a) For the curve $C = \partial D$ oriented counter-clockwise, directly evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$. Hint: to speed things up, have each group member focus on one side of C.

   (b) Now compute $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA$.

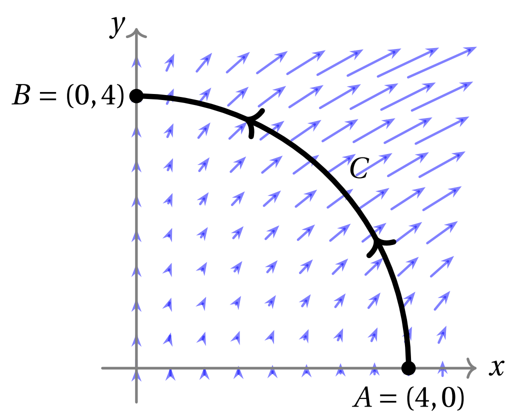
   (c) Check that Green's Theorem works in this example.



2. Compute the line integral of $\mathbf{F}(x,y) = \langle x^3,\ 4x\rangle$ along the path C shown at right against a grid of unit-sized squares. To save work, use Green's Theorem to relate this to a line integral over the vertical path joining B to A. Hint: Look at the region D bounded by these two paths. Check your answer with the instructor.



3. Consider the quarter circle C shown below and the vector field $\mathbf{F}(x,y) = \langle 2xe^y,\ x + x^2e^y\rangle$. The goal of this problem is to compute the line integral $I_0 = \int_C \mathbf{F}\cdot d\mathbf{r}$.



(a) Parameterize C and start directly expanding out I_0 into an ordinary integral in t until you are convinced that finding I_0 this way will be a highly unpleasant experience.

(b) Check that $\mathbf{F}$ is *not* conservative, so we can't use that trick directly to compute I_0.

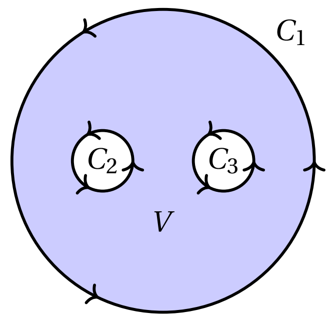
(c) Find a function $f(x, y)$ such that $\mathbf{F} = \mathbf{G} + \nabla f$, where $\mathbf{G}$ is the vector field $\langle 0,\ x\rangle$.

(d) Argue geometrically that $\mathbf{G}$ integrates to 0 along any line segment contained in either the x-axis or the y-axis.

(e) Use part (d) with Green's Theorem to show that $\int_C \mathbf{G} \cdot d\mathbf{r} = 4\pi$.

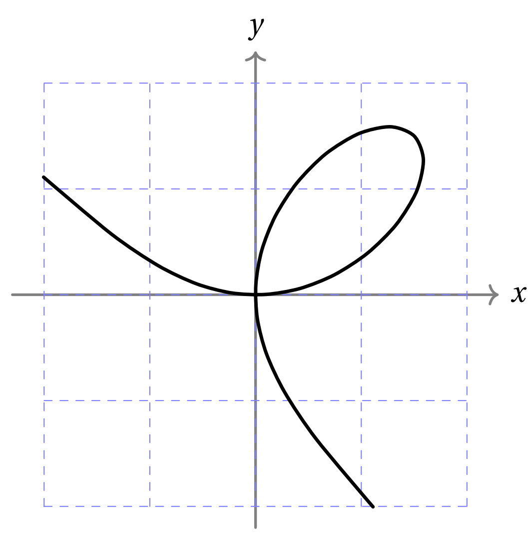
(f) Combine parts (c–e) with the Fundamental Theorem of Line Integrals to evaluate I_0. Check your answer with the instructor.

4. Consider the shaded region V shown, bounded by a circle C_1 of radius 5 and two smaller circles C_2 and C_3 of radius 1. Suppose $\mathbf{F}(x, y) = \langle P,\ Q\rangle$ is a vector field where $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 2$ on V. Assuming in addition that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 3\pi$ and $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = 4\pi$, compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$. Check your answer with the instructor.



5. Suppose D is a region in the plane bounded by a closed curve C. Use Green's Theorem to show that both $\int_C x\ dy$ and $-\int_C y\ dx$ are equal to Area(D).

6. The curve satisfying $x^3 + y^3 = 3xy$ is called the *Folium of Descartes* and is shown at right.



(a) Let C be the "bulb" part of this folium, more precisely, the part in the positive quadrant. Show that any line $y = tx$ for $t > 0$ meets C in exactly two points, one of which is the origin. Use this fact to parameterize C by taking the slope t as the parameter.

(b) Use part (a) and Problem 5 to compute the area bounded by C. Check your answer with the instructor.