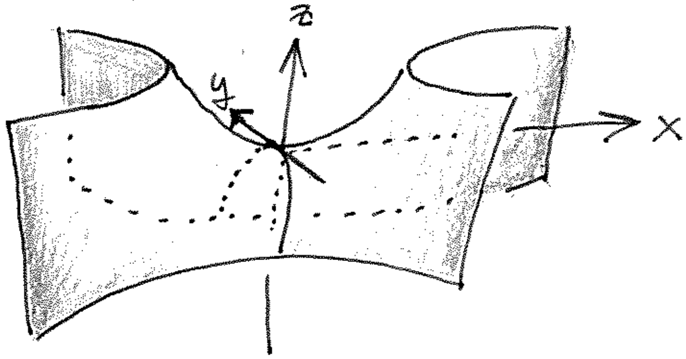
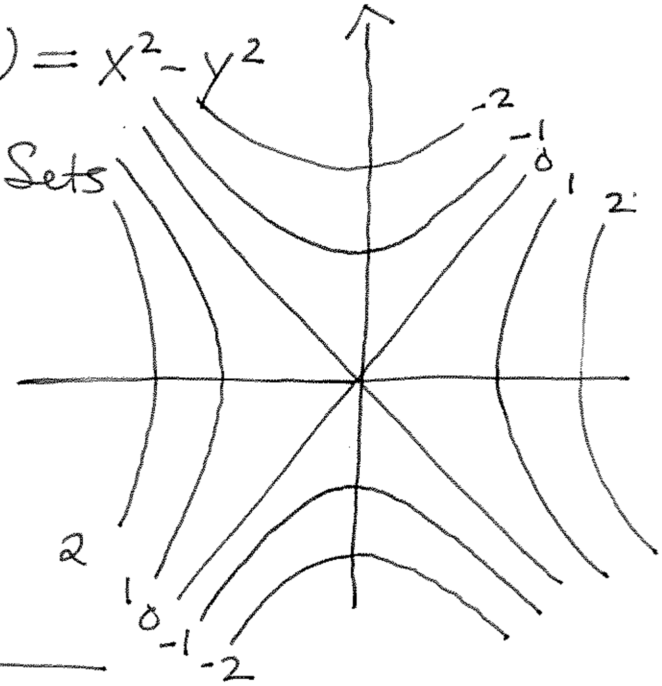
Lecture 6: Level sets in 3^d (§14.1) and quadric surfaces (§12.6); review of limits (§14.2)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$

Graph



Level Sets



For $f: \mathbb{R}^3 \to \mathbb{R}$, can't draw the graph (its in $\mathbb{R}^4$) but can still look at level sets.

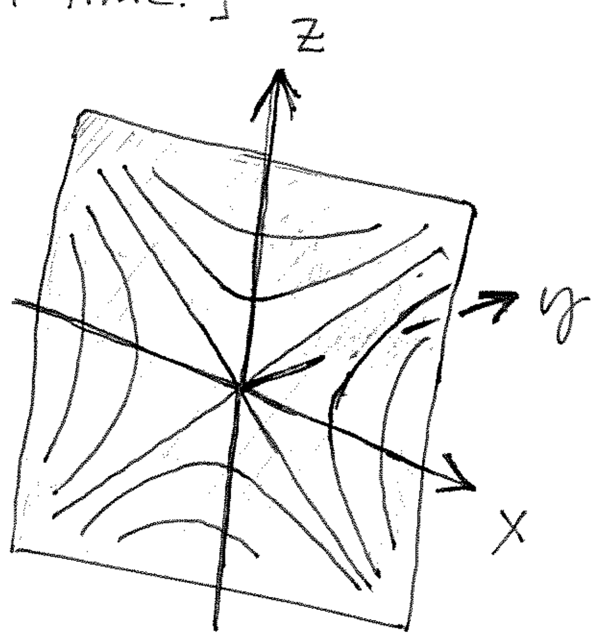
[Did $f(x,y,z) = x^2+y^2+z^2$ last time.]

Ex: $f(x,y,z) = x^2 + y^2 - z^2$

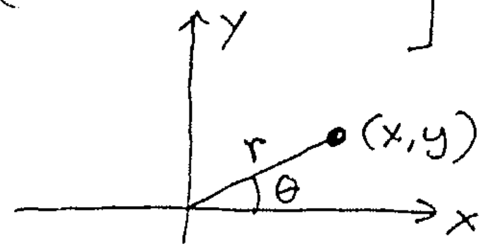
First, in the xz-plane

$$f(x,0,z) = x^2 - z^2$$

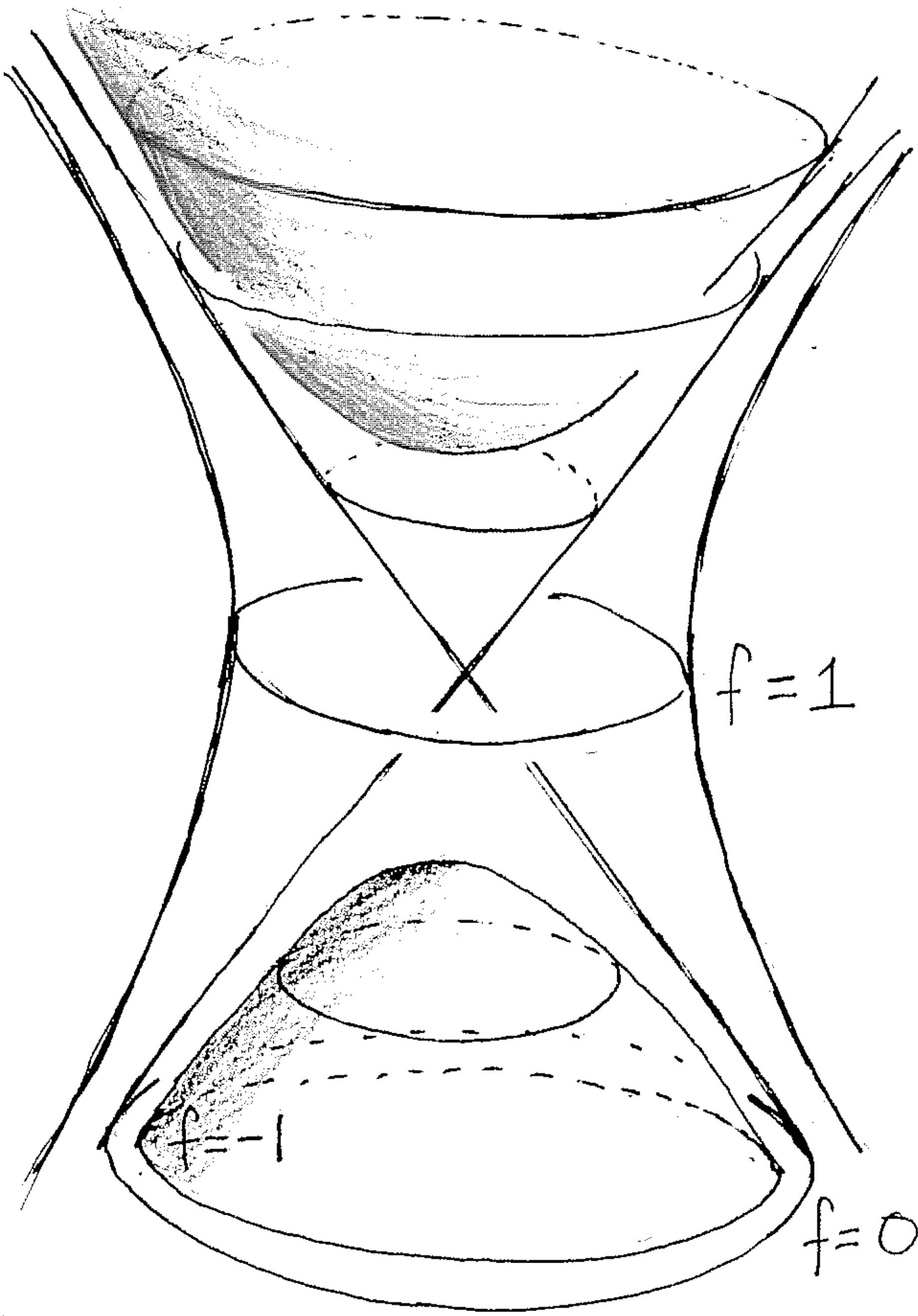
so the level sets there match the above picture.



[Another important tool for drawing graphs and level sets is:]



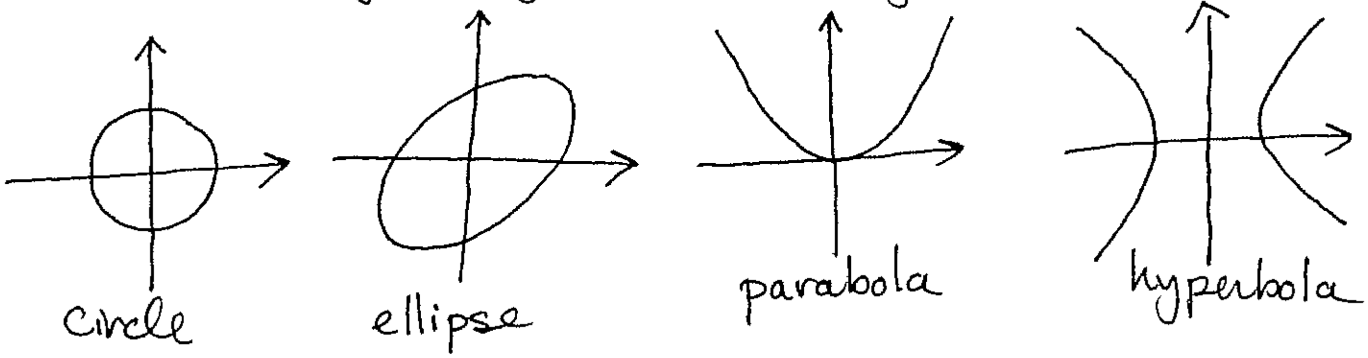
__Symmetry__: As $r^2 = x^2 + y^2$, we can write $f(x,y,z) = r^2 - z^2$. Thus, each level set is rotationally symmetric about the z-axis.



[These level sets are examples of quadric surfaces.]

Conic sections: Solution is $\mathbb{R}^2$ of

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

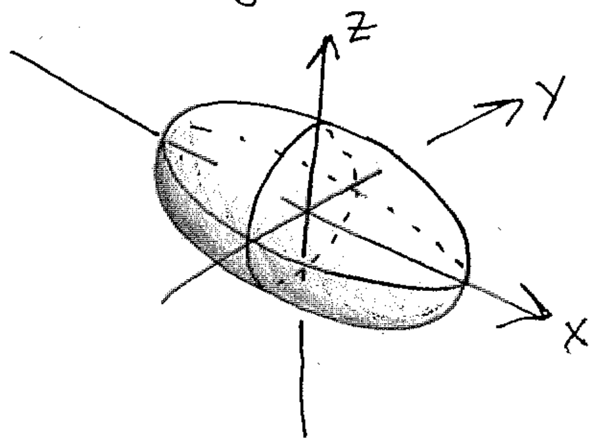


Quadratic surfaces in $\mathbb{R}^3$. (§12.6 and upcoming worksheet)

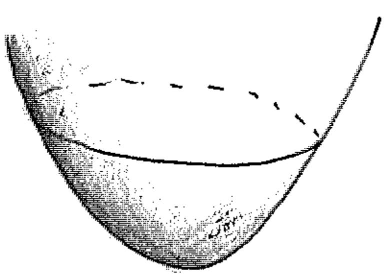
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Ex: Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

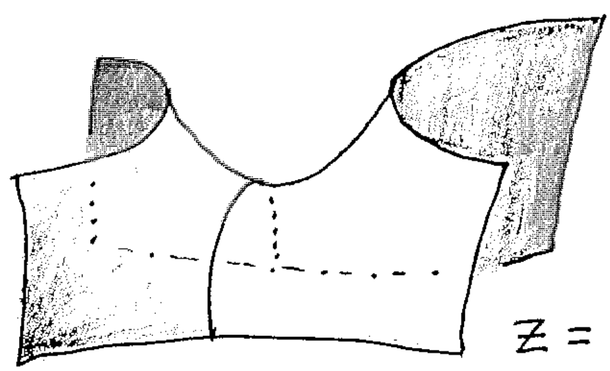


Elliptic paraboloid:



$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

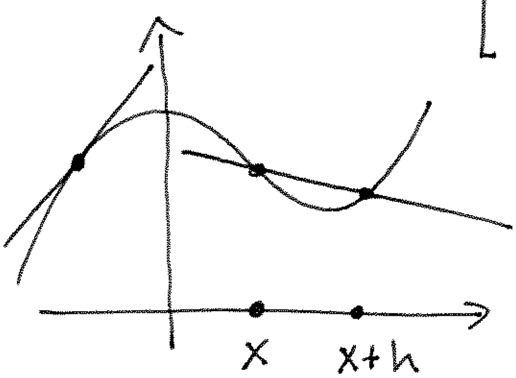
Hyperbolic paraboloid:



$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

[The other quadric surfaces are the (double) cone and the hyperboloids of one and two sheets. You'll learn more about these in section and on the HW.]

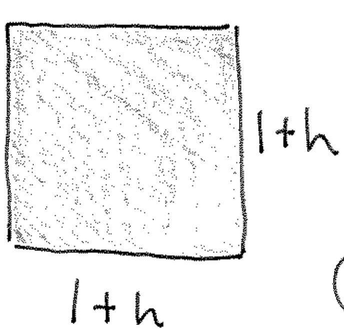
<u>Limits</u> (§14.2) [To talk about derivatives of functions of several variables, we need to understand limits.]



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

[There are different perspectives on limits; I'll focus on them as a way of estimating/controling error.]

Suppose we want to fabricate a square with area $1\,m^2$ but it comes back with sides of length $1+h$

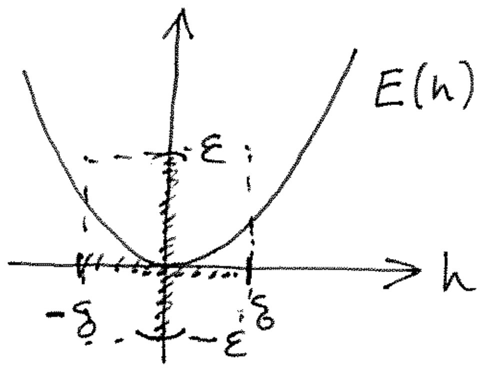


$$\text{Error in area} = (1+h)^2 - 1 = h^2 + 2h$$

<u>Q</u>: If we want this error to be $< 1/10$, to what tolerance do we need to make the square?

Consider $E : \mathbb{R} \to \mathbb{R}$ (think "error function").

We say $\lim_{h \to 0} E(h) = 0$ if given $\varepsilon > 0$ we can always find $\delta > 0$ so that whenever $0 < |h| < \delta$ we have $|E(h)| < \varepsilon$.



<u>Ex</u>: $E(h) = h^2$

[View as challenge-response process.]

<u>1st Challenge</u>: $\varepsilon = 1/10$. Take $\delta = 1/4$. If $|h| < \delta = 1/4$ then $|E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10}$.

<u>2nd Challenge</u>: $\varepsilon = 1/100$ $\quad \delta =$ $\boxed{\text{Audience response}}$

<u>3rd Challenge</u>: $\varepsilon = 1/10000$ $\quad \delta =$ ——— " ———

Claim: $\lim_{h\to 0} h^2 = 0$.

Reason: If you give me $\varepsilon > 0$, I'll take $\delta = \sqrt{\varepsilon}$.
Then if $|h| < \delta$ we have $|h^2| = |h|^2 < \delta^2 = \varepsilon$,
as desired.

Ex: $E(h) = h^2 + 2h$ Know $\lim_{h\to 0} h^2 + 2h = 0$
by "limit laws" from Calc I.

Given $\varepsilon = 1/10$, take $\delta = 1/100$. If $|h| < \delta$,
then $|h^2 + 2h| \leq |h|^2 + 2|h| < \left(\frac{1}{100}\right)^2 + \frac{2}{100}$

$$< \frac{3}{100} < \frac{1}{10} = \varepsilon.$$

Note: In general, say $\lim_{x\to a} f(x) = c$

if $f(a+h) = c + E(h)$

where $\lim_{h\to 0} E(h) = 0$.