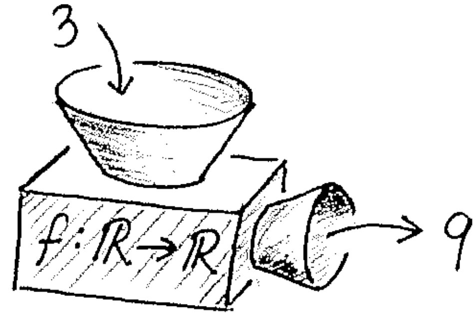
Lecture 5: Functions of Several Variables (§14.1)

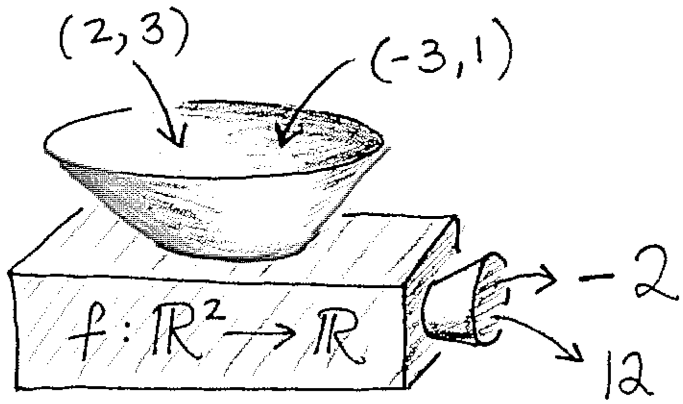
Function of one variable

$$f(x) = x^2$$

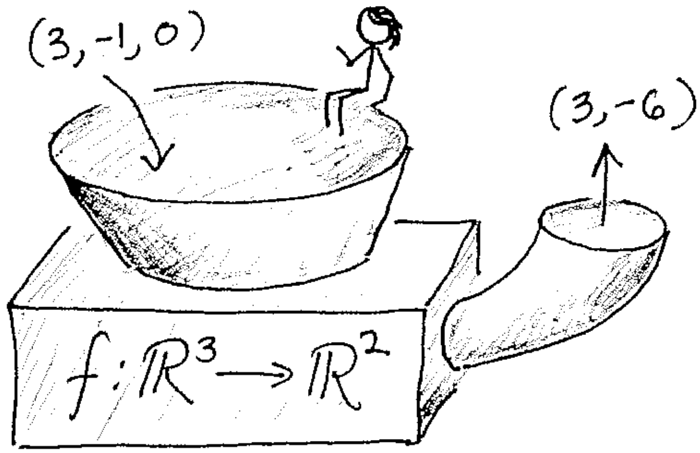


Function of two variables

$$f(x,y) = x^2 - xy$$

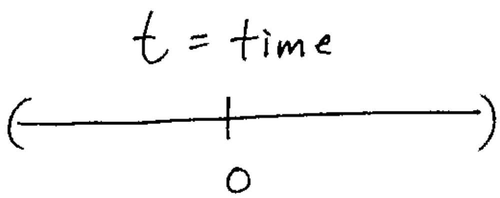


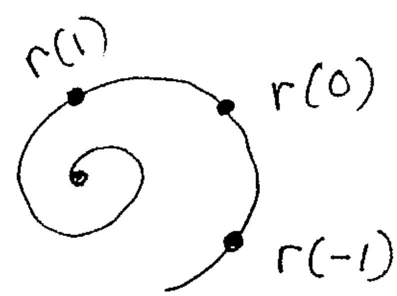
In general, will consider $f: \mathbb{R}^n \to \mathbb{R}^m$.



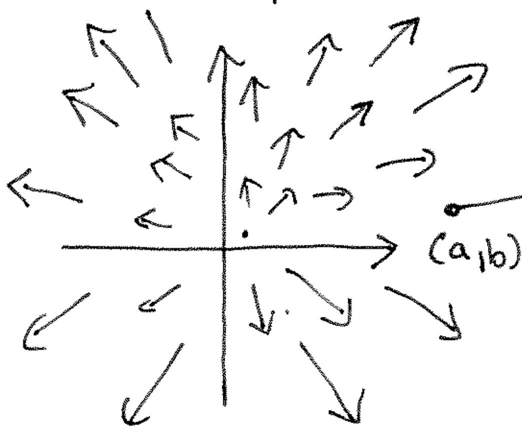
Ex: ① Temperature in this room $T: \mathbb{R}^3 \to \mathbb{R}$

② Parameterized curve in the plane $r: \mathbb{R} \to \mathbb{R}^2$





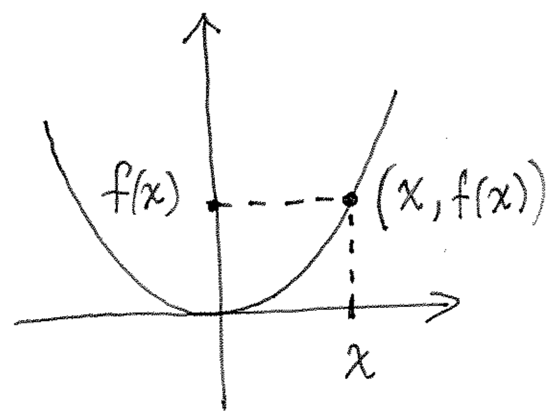
3 Windspeed / direction on a map $f: \mathbb{R}^2 \to \mathbb{R}^2$



[Electric field, force due to gravity, etc.]

[Chapter 14 is 1, Chapter 13 is 2, and Chapter 16 includes 3.]

<u>Today</u>: Functions $\mathbb{R}^2 \to \mathbb{R}$.

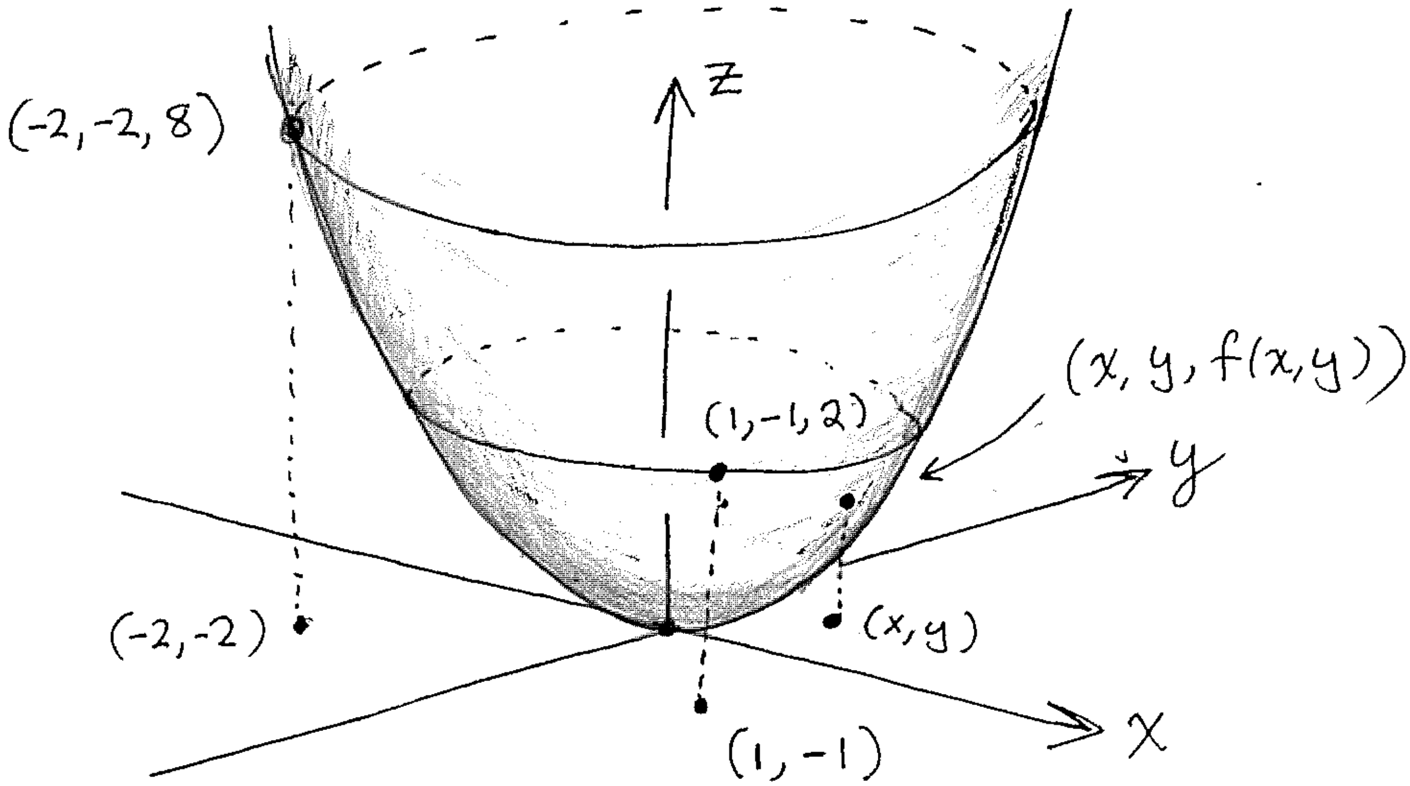


<u>Graphs</u>: For one var $f: \mathbb{R} \to \mathbb{R}$ such as $f(x) = x^2$ have:

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + y^2$.

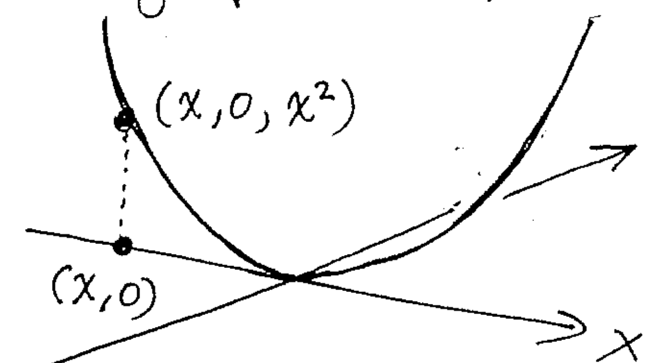
Its graph is $\{(x, y, f(x,y))\}$ in $\mathbb{R}^3$.

and is shown on the next page.



<u>How to figure out</u>: Intersect graph with planes.

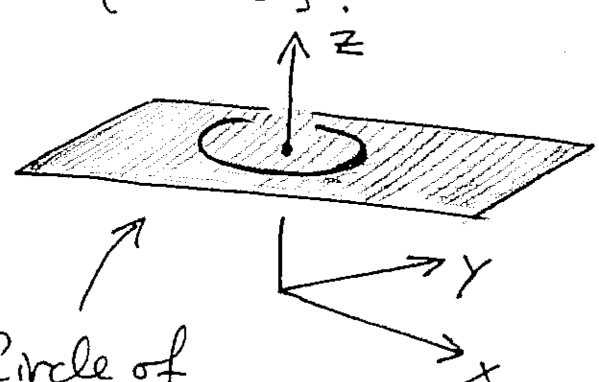
What is over the x-axis? (or y-axis.)



What is the intersection with $\{z = c\}$?

Same as finding all (x, y) with $f(x,y) = c$,

that is, $x^2 + y^2 = c$



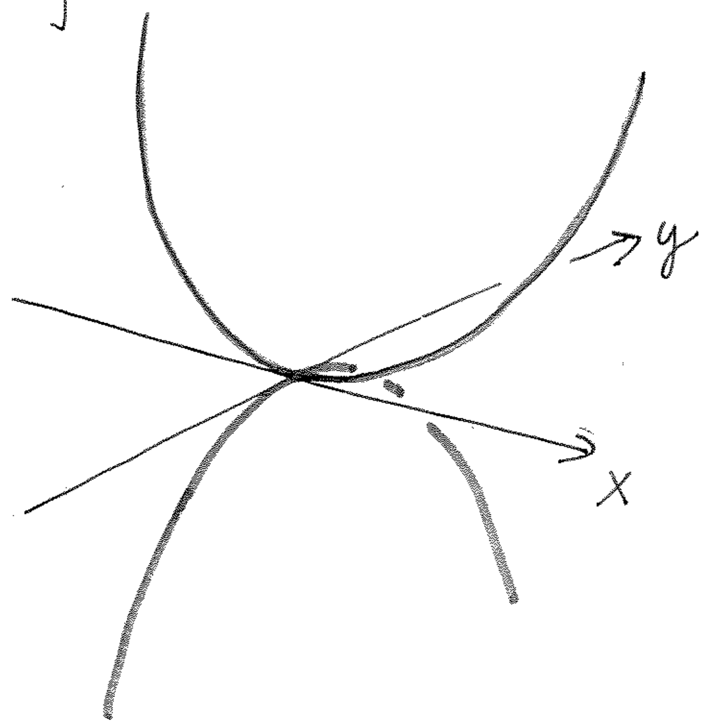
Circle of radius $\sqrt{c}$.

Other tools: symmetry, computers.

Ex: $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x,y) = x^2 - y^2$

$$\left.\begin{array}{l}\text{Over } x\text{-axis: } f(x,0) = x^2 \\ \text{Over } y\text{-axis: } f(0,y) = -y^2\end{array}\right\} \text{Both parabolas}$$

Intersections w/ horizontal planes:



$\underline{z=0}$: $x^2 - y^2 = 0$

$\iff x^2 = y^2$

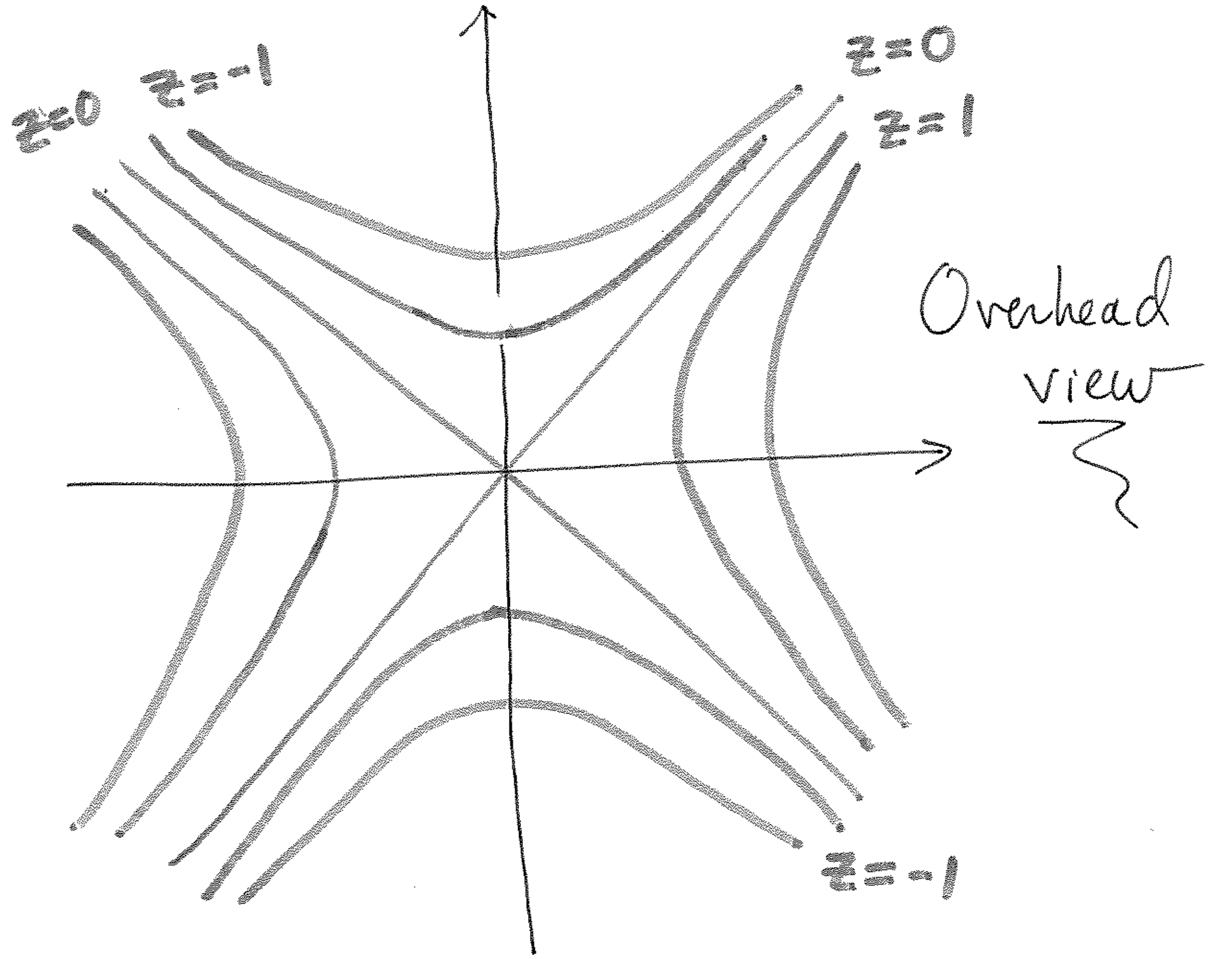
$\iff x = \pm y$

$\underline{z=-1}$: $x^2 - y^2 = -1 \iff y^2 = x^2 + 1$

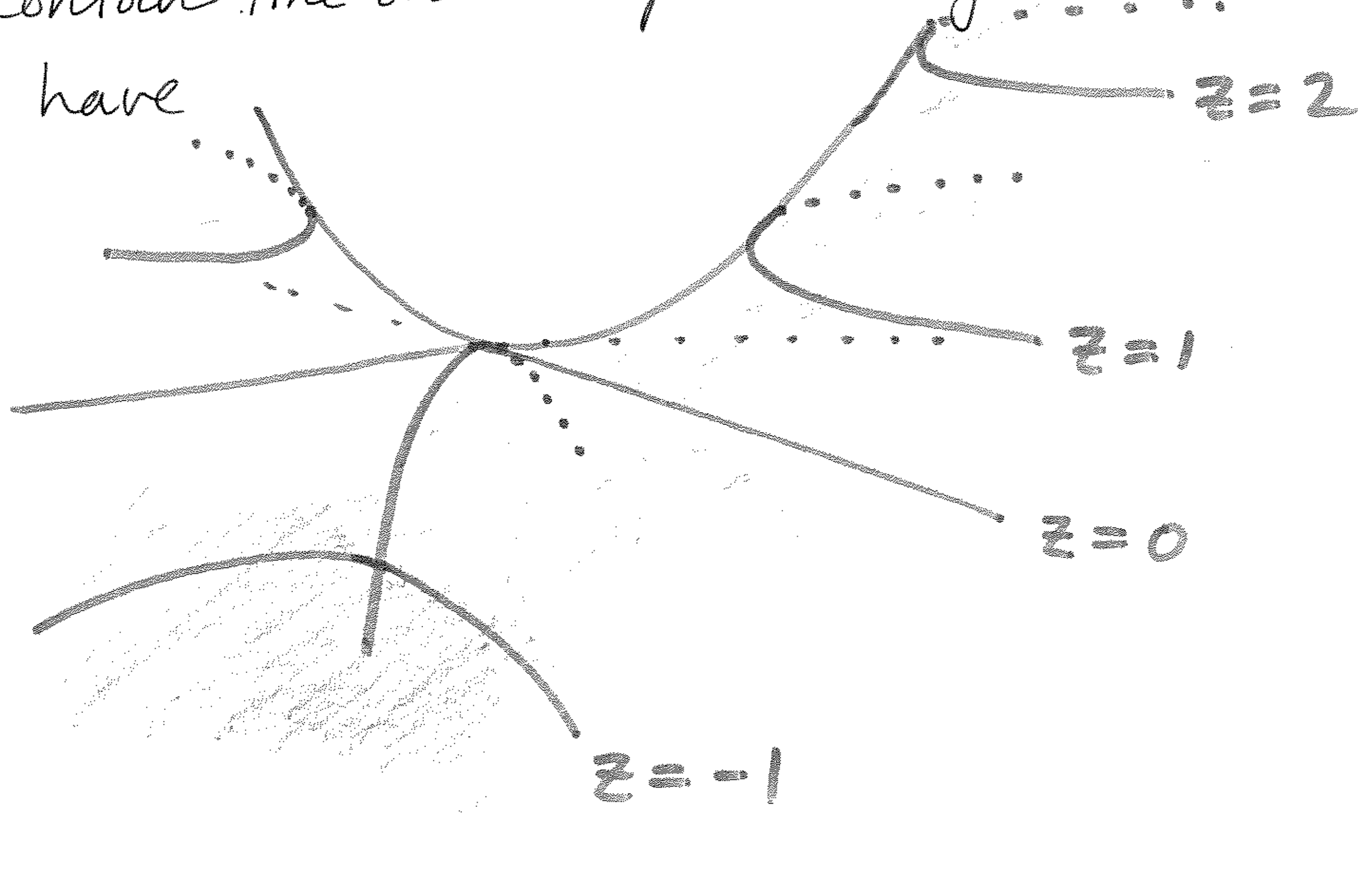
$\iff y = \pm\sqrt{x^2+1}$

$\underline{z=+1}$: $x^2 - y^2 = 1 \iff x = \pm\sqrt{y^2+1}$

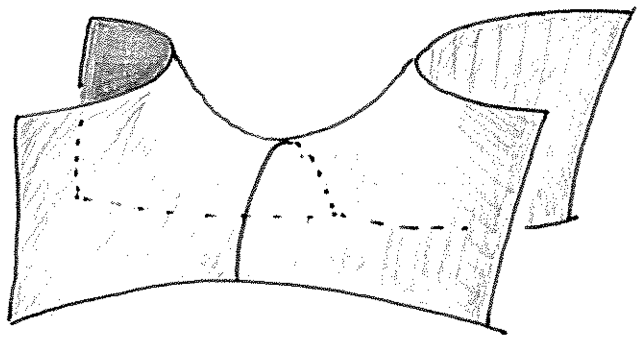
See next page for plot.



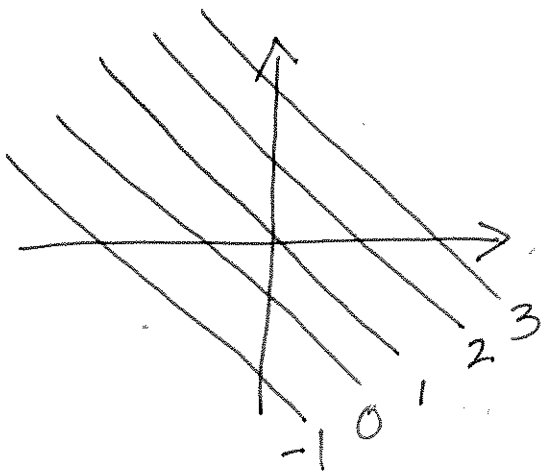
Each of these curves is a level set, like a contour line on a map. Put together, we have



In other words, the graph is a saddle:



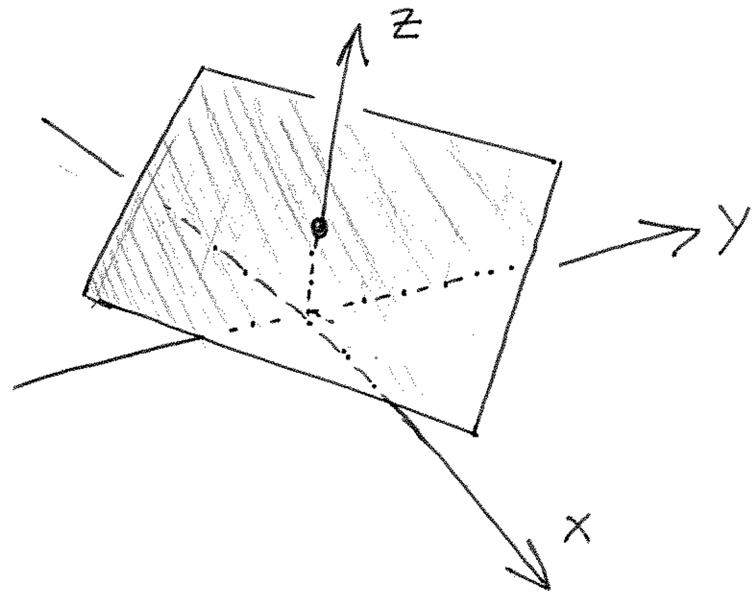
Another example: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x+y+1$



Graph is $z = f(x,y)$,
that is

$$x + y - z + 1 = 0$$

and hence is a plane

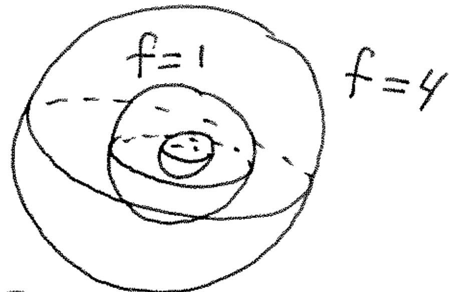


Now, let's try more variables!

<u>Ex</u>: $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x,y,z) = x^2+y^2+z^2$

No graph [well, there is one in $\mathbb{R}^4$.]
but we still have level sets.

$$f = 1 \iff x^2+y^2+z^2 = 1$$



<u>Ex</u>: Hopf fibration $f: \mathbb{R}^3 \to \mathbb{R}^3$
has image contained in . Level
sets are mostly circles. [Show video.]