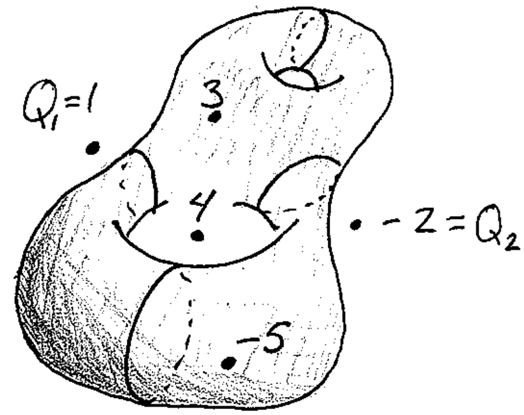
Lecture 38: Maxwell's equations

__Last time:__ Charges Q_i at positions $\vec{p}_i$

$$\vec{E}(\vec{r}) = \sum \frac{Q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{p}_i|^3}(\vec{r}-\vec{p}_i)$$



__Gauss's Law:__ D a region in $\mathbb{R}^3$

$$\iint_{\partial D} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0}\left(\text{Total charge in } D\right)$$

$$\text{Flux} = \frac{-2}{\varepsilon_0}$$

[On HW for today, used this compute the total charge from a formula for $\vec{E}$. This is not impractical... When there are many (e.g. 10^{20}) can't use ~~the~~ the formula for $\vec{E}$ directly. Don't want to focus on each charge individually just as we don't ~~count~~ count molecules when measuring the mass.]

__Mass:__ $\rho(x,y,z)$ mass density, units $= \frac{g}{m^3}$

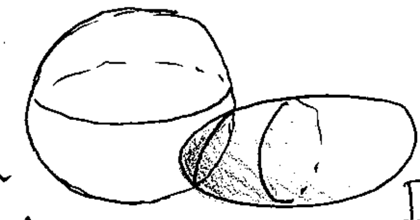
D

$$\text{Total Mass of } D = \iiint_D \rho\, dV$$

__Charge:__ $\rho(x,y,z)$ charge density, units $= \frac{\text{Coulomb}}{m^3}$

$$\text{Total Charge of } D = \iiint_D \rho\, dV$$

Q: How does ρ determine $\vec{E}$?



Gauss's Law should still hold, so for a region R we have

$$\frac{1}{\varepsilon_0}\begin{pmatrix}\text{Charge}\\ \text{in } R\end{pmatrix} = \iint_{\partial R} (\vec{E}\cdot\vec{n})\, dA \underset{\text{Divergence Thm}}{=} \iiint_R \operatorname{div} \vec{E}\, dV$$

$$\| $$

$$\frac{1}{\varepsilon_0}\iiint_R \rho\, dV$$

As true for <u>all</u> regions R, must have $\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$

[<u>Q</u>: Does this answer the question? Not completely since many vector fields have the <u>same</u> divergence.]

<u>A</u>. $\vec{E}(\vec{r}) = (E_1(\vec{r}), E_2(\vec{r}), E_3(\vec{r}))$ where if $\vec{r} = (a,b,c)$

then

$$E_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_D \frac{(a-x)\,\rho(x,y,z)}{|\vec{r}-(x,y,z)|^3}\, dV$$

etc.

<u>Exercise</u>: Take D =  unit sphere and $\rho = 1$.
Use above to calculate the electric field at $\vec{r} = (a,b,c)$

<u>Hint</u>: Use symmetry to reduce to the case where $\vec{r} = (a,0,0)$.

Maxwell's Equations:

$\vec{E}(x,y,z,t)$ – Electric field (at time t) $(\mathbb{R}^4 \to \mathbb{R}^3)$

$\vec{B}(x,y,z,t)$ – Magnectic field

$\rho(x,y,z,t)$ – charge dnsity $(\mathbb{R}^4 \to \mathbb{R})$

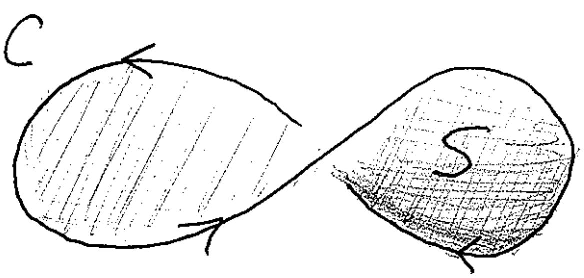
Gauss's Law:

$$\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \iint_{\partial R} \vec{E} \cdot \vec{n}\, dA = \iiint_R \rho \, dV$$

Gauss's Law for magnetic fields: [No magnectic monopoles.]

$$\operatorname{div} \vec{B} = 0 \qquad \iint_{\partial R} (\vec{B} \cdot \vec{n})\, dA = 0$$

Faraday's Law of Induction: A changing magnetic field induces a current in a loop of wire.



$$\underbrace{\int_C \vec{E} \cdot d\vec{r}}_{\substack{\text{electromotive force} \\ \text{aka voltage}}} = -\frac{\partial}{\partial t} \iint_S \vec{B} \cdot \vec{n}\, dA$$

Now

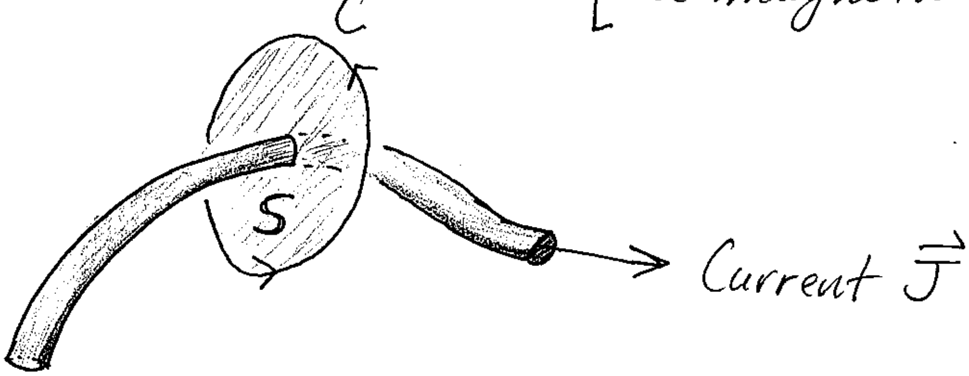
$$\int_C \vec{E} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{E}) \cdot \vec{n}\, dA \quad \text{and so} \quad \boxed{\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Unit check: (a) $\vec{F} = Q\vec{E} \Rightarrow \vec{E}$ in $\frac{N}{C} = \frac{V}{m} \Rightarrow \int_C \vec{E}\cdot \underset{m}{d\vec{r}}$ is in volts.

(b) $\vec{B}$ has units $T = \text{Tesla} = \frac{Vs}{m^2}$ $\quad \vec{n} = $ unitless

So $\iint_S \vec{B}\cdot\vec{n}\, \underbrace{dA}_{m^2}$ is in $Vs \Rightarrow \frac{\partial}{\partial t}\iint_S (\vec{B}\cdot\vec{n})\, dA$ is in V.

Ampere's circuital law: [Current in a wire or a changing electric field induces a magnetic field.]



$$\int_C \vec{B}\cdot d\vec{r} = \mu_0 \iint_S (\vec{J}\cdot\vec{n})\, dA + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\iint_S (\vec{E}\cdot\vec{n})\, dA$$

$$\parallel$$

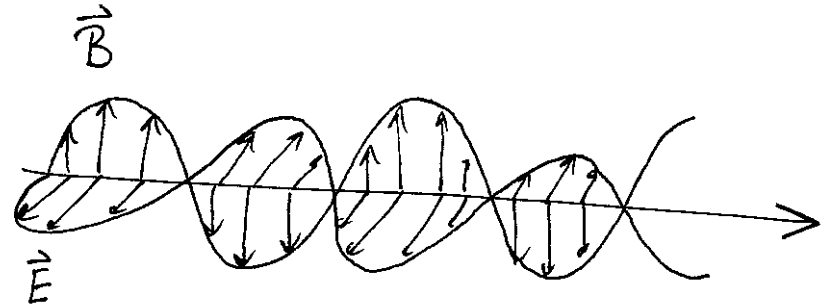
$$\iint_S (\text{curl}\,\vec{B})\cdot\vec{n}\, dA$$

So $\boxed{\text{curl}\,\vec{B} = \mu_0 \vec{J} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}}$

$\varepsilon_0 =$ permittivity of free space $=$ farads/m

$\mu_0 =$ permeability of free space $= N/A^2$

And there was light...



$$C = \text{speed of light} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$C = \underset{\text{freq}}{\omega}\ \lambda \leftarrow \text{wave length}$$

$$\vec{E} = \left(0,\ C \cdot \cos(2\pi(\omega t - x/\lambda)),\ 0\right)$$

$$\vec{B} = \left(0,\ 0,\ \cos(2\pi(\omega t - x/\lambda))\right)$$

Check: $\operatorname{curl} \vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\qquad \operatorname{curl} \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

$\operatorname{div} \vec{E} = 0$ $\qquad \operatorname{div} \vec{B} = 0.$

[**Moral**: Here are other applications of line and surface integrals, as well as the relations between them.]

If time remains: ⓐ Multitude of integral theorems.

ⓑ $$\int_{\partial M} \omega = \int_M d\omega$$

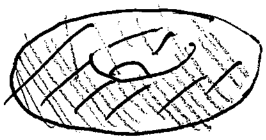
The general Stokes' Thm.

n-dim'l manifold: locally like $\mathbb{R}^n$

n=1: 

n=2: 

$\mathbb{R}^2$

n=3: 

$S^3 = \{x^2+y^2+z^2+w^2=1 \text{ in } \mathbb{R}^4\}$

n=4: $\mathbb{R}^4$ $\quad S^4 = \{\sum_{i=1}^{5} x_i^2 = 1 \text{ in } \mathbb{R}^5\}$

differential n-form: something that can be integrated on an n-manifold

eats n-tangent vectors and computes a "volume".

An n-form ω has a derivative $d\omega$, an $(n+1)$-form.

Stokes's Thm: Suppose M is an oriented $(n+1)$-dim'l manifold and ω an n-form on M. Then

$$\int_{\partial M} \omega = \int_M d\omega$$

[Should really require ω to be compactly supported.]

This subsumes F.T.L.I., Green's Thm, Stokes' Thm, and the Divergence Thm.

For M in $\mathbb{R}^3$, a vector field can be associated to both a $\underbrace{\text{1-form}}_{d \text{ is curl}}$ and a $\underbrace{\text{2-form}}_{d \text{ is div}}$

A 0-form is just a function, d of such is ∇.

Leibniz rule: $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$ if α is a p-form.

$d \circ d = 0 \rightsquigarrow$ cohomology

Modern version of Stokes' Thm due to Cartan in 1940s.