Lecture 36: Conservative vector fields in $\mathbb{R}^3$ (§16.8); Topology 101

Previously on Math 241...

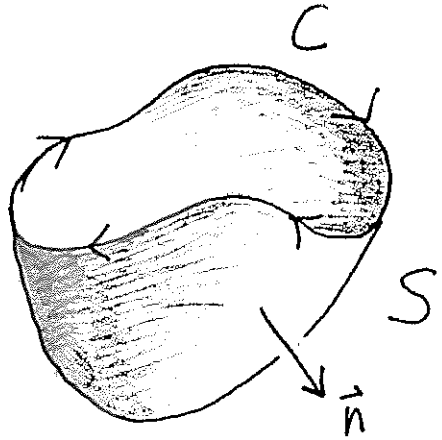
A vector field $\vec{F}$ is conservative when $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \to \mathbb{R}$.

Thm A: A vector field on an open connected region R in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in R.

Thm B′: A vector field $\vec{F}$ on all of $\mathbb{R}^3$ is conservative if and only if $\operatorname{curl} \vec{F} = \vec{0}$ everywhere.

[Explained part of this last time, will rest now.]

Other idea behind Thm B′: Suppose $\operatorname{curl} \vec{F} = \vec{0}$ and C is a closed curve in $\mathbb{R}^3$. [Need $\int_C \vec{F} \cdot d\vec{r} = 0$ so that Thm A applies.] Suppose S is an orientable surface S with $\partial S = C$.



By Stokes:

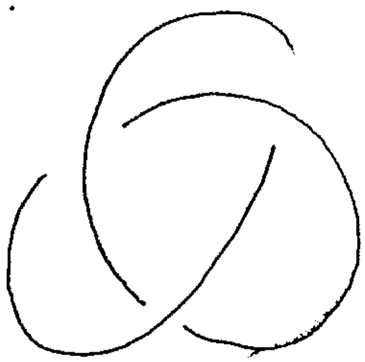
$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\,dA$$

$$= \iint_S \vec{0}\cdot\vec{n}\,dA = 0.$$

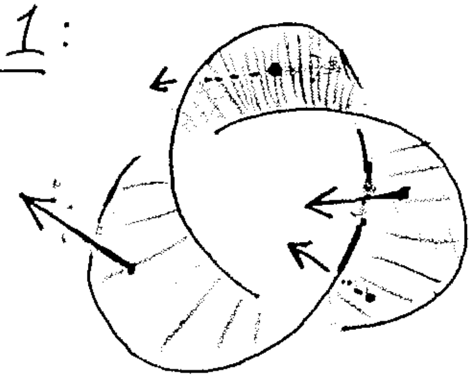
Q: Is every C the boundary of some such S?

If yes, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed C and so Thm A gives that $\vec{F}$ is conservative.

Ex:



Try 1:

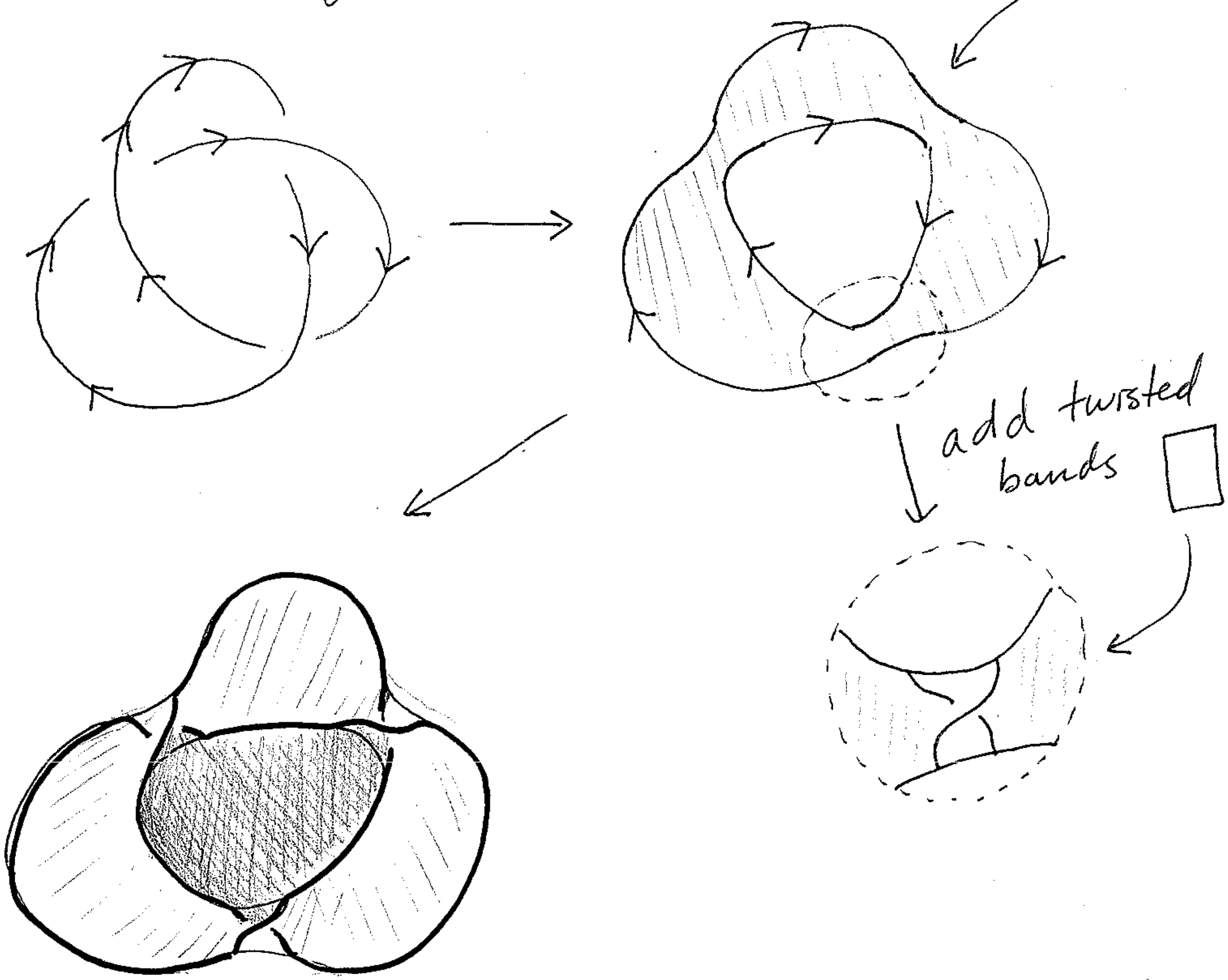


Problem: not orientable.

Fact: There is always such a surface.

In fact, there is always such a surface

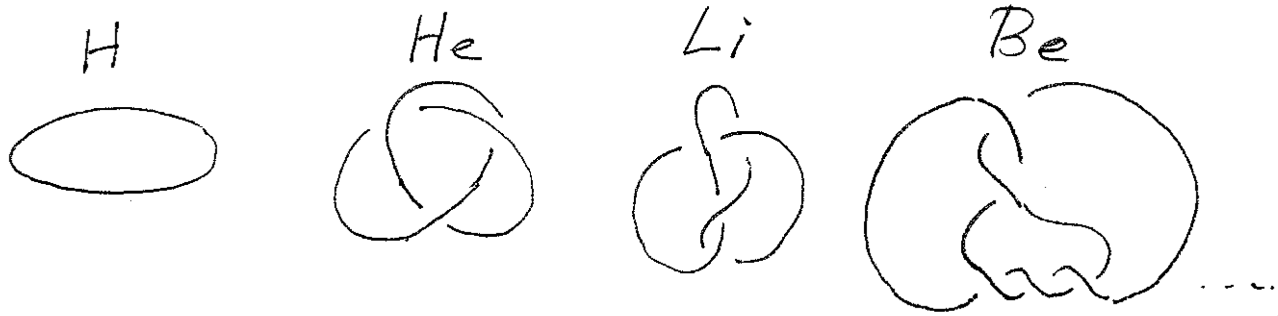
[can be found algorithmically.]



which is actually orientable (normals point at you for both discs).

A closed curve (w/o self intersections) is called a <u>knot</u>. Part of a branch of mathematics called <u>topology</u>.

History: Lord Kelvin: atoms as knots in the "ether"
Tait: made a table of simple knots (1870s)



Actually, theres no ether. (Michleson-Morley 1880s)
Einstein 1905.

Mathematicians thought about knots anyway for 100 years. Now used in biology studying action of enzymes on DNA....

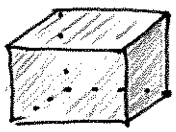
Public key cryptography...

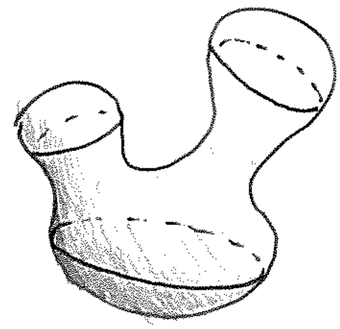
Least area surfaces...

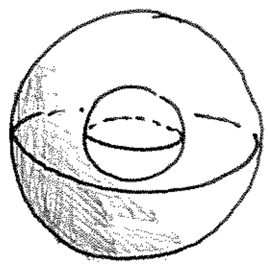
[More generally have...]

<u>Thm B''</u>. A vector field $\vec{F}$ on an open simply connected R in $\mathbb{R}^3$ is conservative if and only if $\operatorname{curl} \vec{F} = \vec{0}$ on R.

Simply connected:

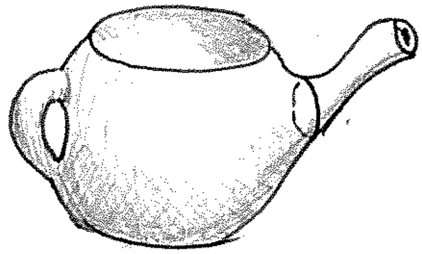






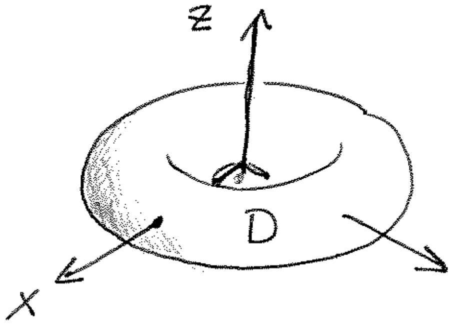
$1 \leq x^2 + y^2 + z^2 \leq 1$

Not simply connected:



Cohomology: Counting holes using vector fields

Consider: The vector field $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$



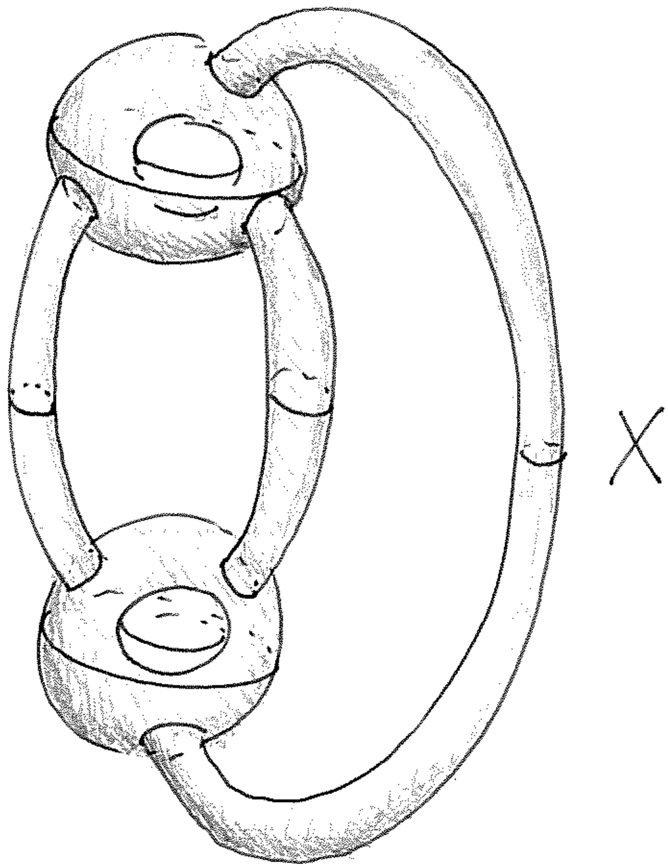
makes sense on D and has curl $\vec{F} = \vec{0}$ but is <u>not</u> conservative. Turns out that if $\vec{G}$ on D has curl $\vec{G} = \vec{0}$ then $\vec{G} = c\vec{F} + \nabla f$ for some c in $\mathbb{R}$ and $f: D \to \mathbb{R}$. So there's basically one irrotational vector that is not conservative $\rightsquigarrow$ D has exactly one "1-dimensional hole".

Count "2-dim holes" by looking at incompressible $\vec{F}$ (i.e. div $\vec{F} = 0$) that aren't equal to curl $\vec{G}$ for any vector field $\vec{G}$.

Surprising Key: $\text{curl}(\nabla f) = \vec{0}$ for any f

$\text{div}(\text{curl}\, \vec{F}) = \vec{0}$ for any $\vec{F}$.

Ex:



Has $\underline{2}$ "1-dim holes"

and $\underline{2}$ "2-dim holes"