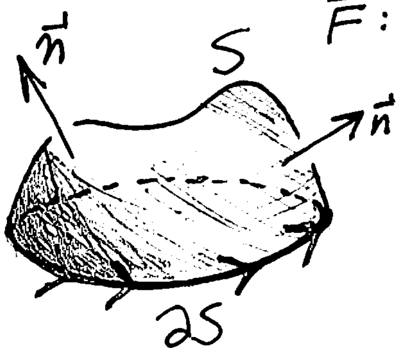
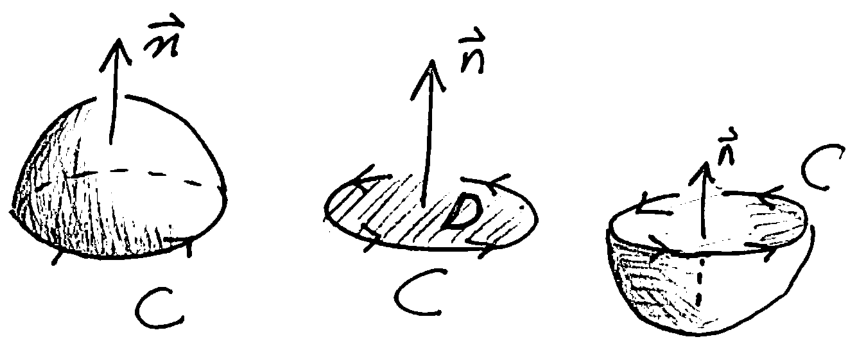
Lecture 35: More on Stokes' Theorem (§16.8), including understanding the curl (§16.5) and the connection to conservative vector fields (§16.8).

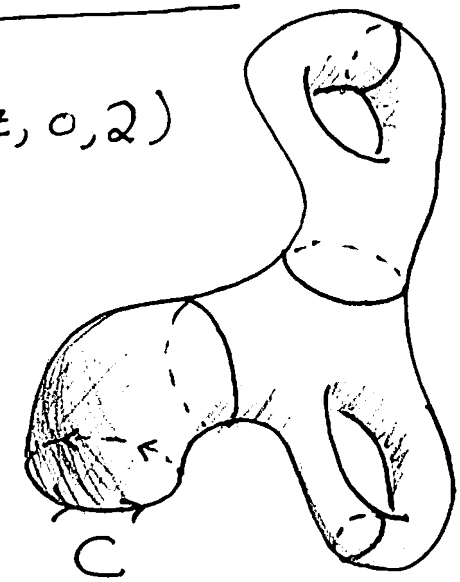
Last time: Stokes' Thm: S surface in $\mathbb{R}^3$
$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ vector field. Then



$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA$$

Ex: $\vec{F} = (-y, x, yz)$ $\qquad$ $\operatorname{curl} \vec{F} = (z, 0, 2)$





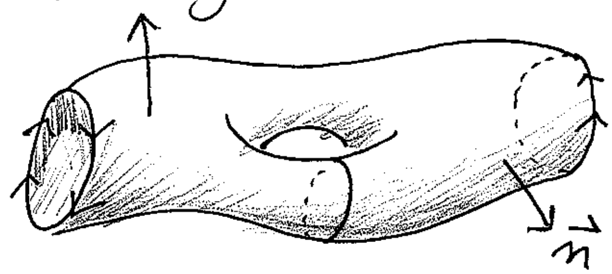
$$\iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA = 2\pi = \int_C \vec{F} \cdot d\vec{r} \text{ for all of these!}$$

[Takes some getting used to, is really just Green's Thm / 2d Divergence Thm in disguise...]

Check the easy one: $\iint_D (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA$

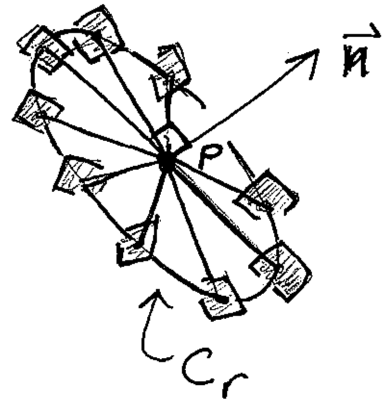
$$= \iint_D (z, 0, 2) \cdot (0, 0, 1) \, dA = \iint_D 2 \, dA = 2 \operatorname{Area}(\bigcirc)$$
$$= 2\pi$$

Note: Also works when S has several boundary components, [provided they are oriented correctly.]



Understanding Curl: Consider a small paddle wheel at p, of radius r.

$\vec{F}$ = fluid flow

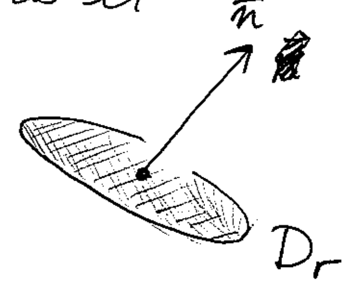


Q: How fast does it spin?

A: $\omega = \dfrac{1}{2\pi r^2} \displaystyle\int_{C_r} \vec{F} \cdot d\vec{r}$

$\boxed{\text{units} = \dfrac{\text{radians}}{\text{time}}}$

Plausible: ⓐ Want tangential component of $\vec{F}$ as it hits the paddles.
ⓑ Looks like an average (almost).



Stokes says:

$$\omega = \frac{1}{2\pi r^2} \iint_{D_r} (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA$$

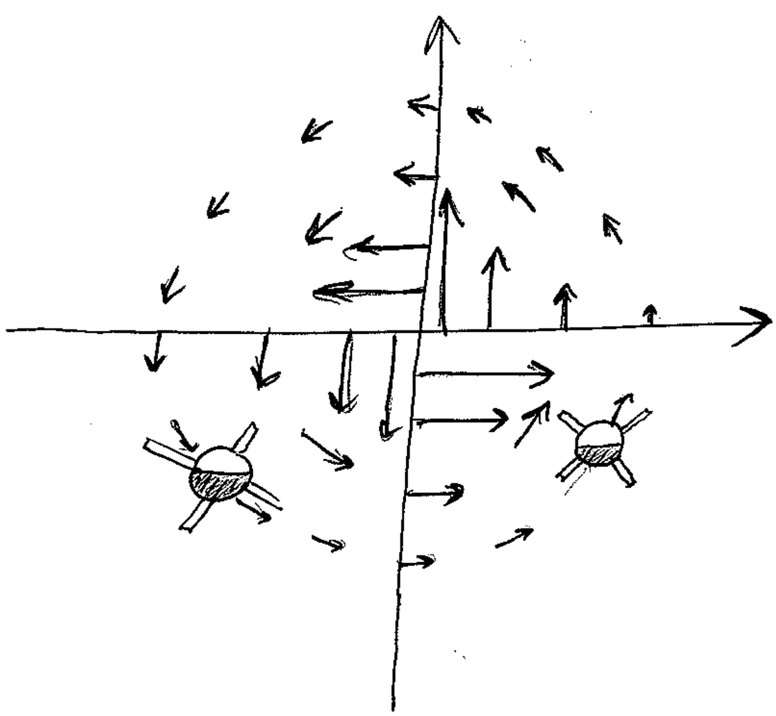
$$= \frac{1}{2}\left(\frac{1}{\text{Area}(D_r)} \iint_{D_r} (\text{curl } \vec{F}) \cdot \vec{n} \; dA \right)$$

Taking $r \to 0$ get: $\omega = \frac{1}{2}(\text{curl } \vec{F}) \cdot \vec{n}$

So the rate of rotation is fastest the direction of $\text{curl } \vec{F}(p)$ and then $\omega = \frac{1}{2}|\text{curl } \vec{F}|$.

<u>Note</u>: A vector field where $\text{curl } \vec{F} = \vec{0}$ everywhere are called <u>irrotational</u>.

<u>Ex</u>: $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ has $\text{curl } \vec{F} = \vec{0}$ except at $(0,0)$ where it's not defined.



Experimentally, a draining tub is an irrotational flow!

<u>Conservative Vector Fields.</u> Recall $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$.

<u>Thm B</u>: Suppose $\vec{F} = (P, Q)$ is a vector field on an open simply-connected region D in $\mathbb{R}^2$. Then $\vec{F}$ is conservative if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ on D.

[Want this kind of easy to use test for $\mathbb{R}^3$.]

<u>Thm B′</u>: A vector field $\vec{F}$ defined on all of $\mathbb{R}^3$ is conservative if and only if $\operatorname{curl} \vec{F} = \vec{0}$ everywhere.

<u>Note</u>: Example on last page has $\operatorname{curl} \vec{F} = \vec{0}$ but is <u>not</u> conservative because of:

<u>Thm A</u>: A vector field $\vec{F}$ on an open connected region R in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in R.

Compatible with Thm B′ since $\vec{F}$ is not defined at $(0, 0, 0)$.

Idea behind Thm B$'$, part I. Suppose $\vec{F}$ is conservative, with $\vec{F} = \nabla f$ for some $f : \mathbb{R}^3 \to \mathbb{R}$. Then

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \Big(\underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial f^2}{\partial z \partial y}}_{=0},\ 0,\ 0 \Big)$$

and so $\operatorname{curl} \vec{F} = \vec{0}$ everywhere.

Next time: I'll explain why $\operatorname{curl} \vec{F} = \vec{0}$ means $\vec{F}$ is conservative.