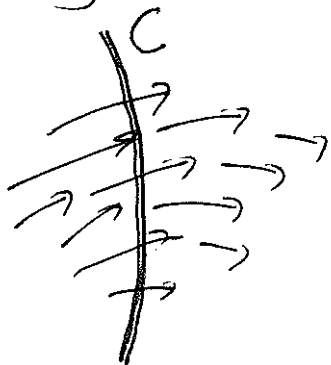


Lecture 33: Surface integrals of vector fields and the Divergence Theorem in 3D (§16.7 and §16.9)

Previously: $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ Flux = rate fluid crosses C

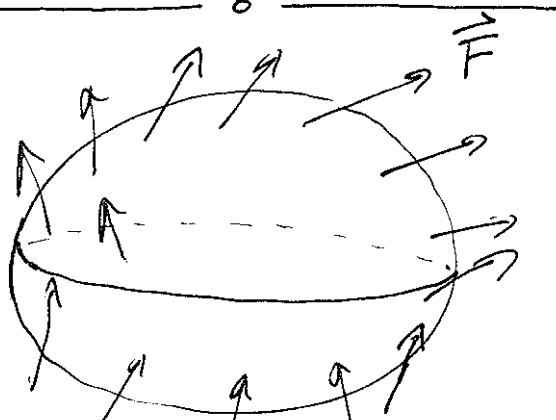


$$= \int_C (\vec{F} \cdot \vec{n}) ds$$

where \vec{n} is a unit normal vector field.

$$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

S a surface

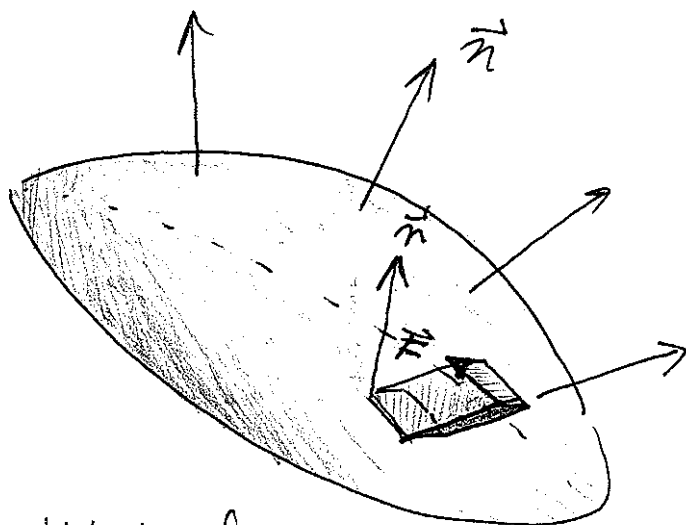
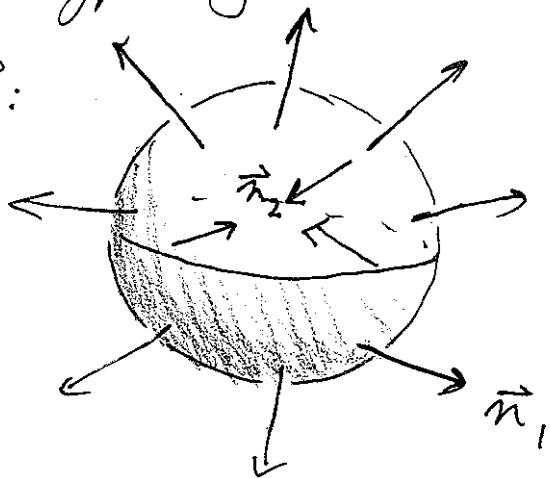


$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n} dA = \iint_S \vec{F} \cdot d\vec{S}$$

↑
Notation

where \vec{n} is a unit normal vector field.

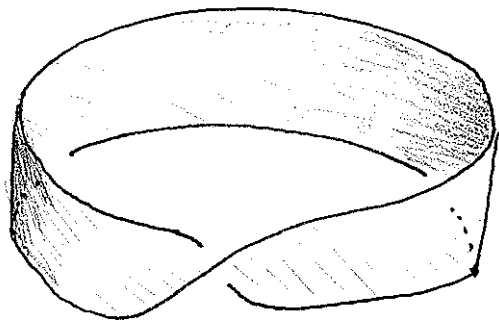
Note: Typically two choices for \vec{n} :



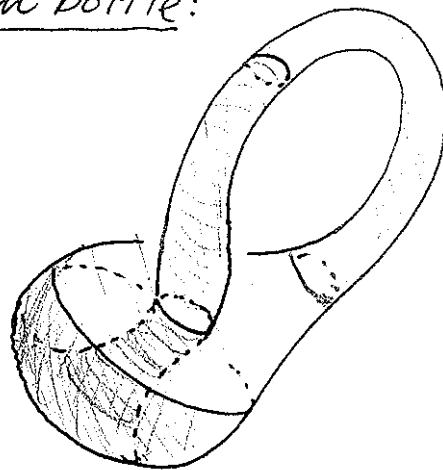
Works for same reason as before.

Some surfaces have no good choice of normal vector,
 so can't integrate vector fields on these:

Möbius band:



Klein bottle:



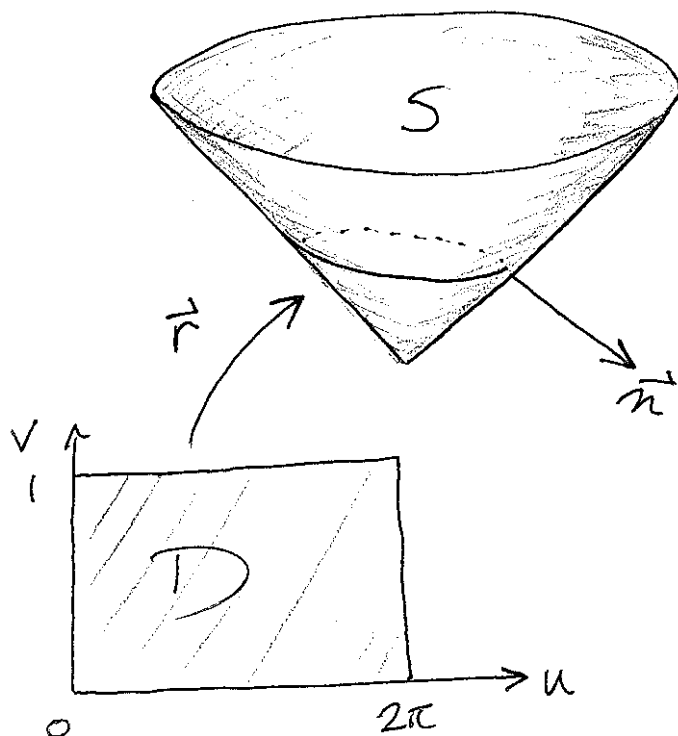
[Show physical models.]

Ex: Cone $x^2 + y^2 = z^2$ $0 \leq z \leq 1$

$$\vec{F} = (z, 1, x)$$

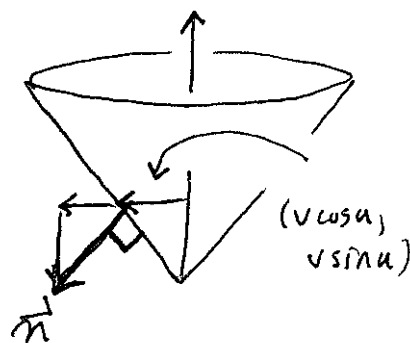
Compute: $\iint_S (\vec{F} \cdot \vec{n}) dA$

$$\vec{r}(u, v) = (v \cos u, v \sin u, v)$$



$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{(v \cos u, v \sin u, -v)}{\sqrt{v^2 \cos^2 u + v^2 \sin^2 u + v^2}} \leftarrow \text{from earlier}$$

$$= \frac{1}{\sqrt{2}} (\cos u, \sin u, -1)$$



$$\underbrace{\vec{F}(\vec{r}(u,v)) \cdot \vec{n}} = \frac{1}{\sqrt{2}} (v \cos u + \sin u - v \cos u)$$

$$(v, 1, v \cos u) = \frac{1}{\sqrt{2}} \sin u$$

$$S_0 \iint_S (\vec{F} \cdot \vec{n}) dA = \int_0^1 \int_0^{2\pi} \left(\frac{1}{\sqrt{2}} \sin u\right) \cdot \overbrace{|\vec{r}_u \times \vec{r}_v|}^{\sqrt{2}v} du dv$$

$$= \int_0^1 \int_0^{2\pi} v \sin u du dv = 0$$

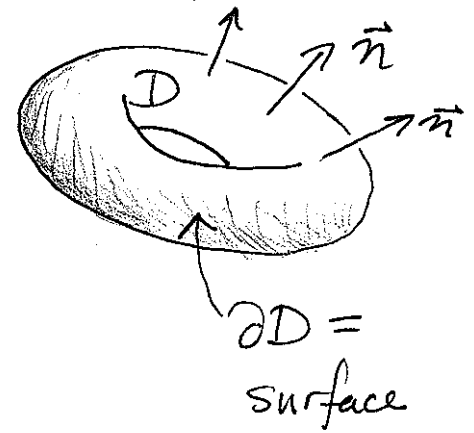
Short cut:

$$\iint_S (\vec{F} \cdot \vec{n}) dA = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \cdot |\vec{r}_u \times \vec{r}_v| du dv$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

Divergence Thm: D a region in \mathbb{R}^3 , \vec{F} a vector field
 \vec{n} outward pointing normal.

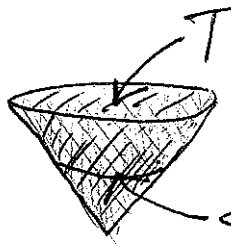
$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \iiint_D \text{div } \vec{F} dV$$



Here, if $\vec{F} = (F_1, F_2, F_3)$ then

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \left(\begin{array}{l} \text{a fn from} \\ \mathbb{R}^3 \rightarrow \mathbb{R} \end{array} \right)$$

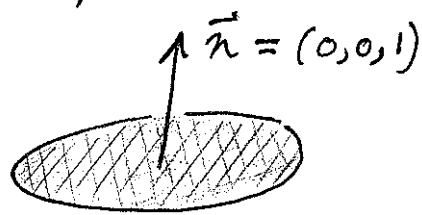
Everything can be interpreted as before.

Ex: $D =$  = solid = $z^2 \geq x^2 + y^2$
 $z \leq 1$

$$\vec{F} = (z, 1, x)$$

$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \underbrace{\iint_S (\vec{F} \cdot \vec{n}) dA}_0 + \iint_T (\vec{F} \cdot \vec{n}) dA$$

$$\iint_T (\vec{F} \cdot \vec{n}) dA = \iint_T x dA = 0 \quad \text{by symmetry.}$$



$$\text{So } \iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \boxed{0}$$

Compare:

$$\iiint_D \operatorname{div} \vec{F} dV = \iiint_D 0 dV = \boxed{0} \quad \checkmark$$

$$\frac{\partial}{\partial x} z + \frac{\partial}{\partial y} 1 + \frac{\partial}{\partial z} x = 0$$

Application: Heat flow.

$u(x, y, z, t)$ = temperature at (x, y, z) at time t .

$$\frac{\partial u}{\partial t} = c \Delta u = c \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

↑ Laplacian

Will now explain why.

Newton's Law of Cooling:

$$\text{Heat flow} = -k \text{ grad}(u) = -k \overbrace{\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right)}^{\text{hot to cold}}$$

Rate heat flows into a region D :

$$\iint_{\partial D} (-k \text{ grad}(u)) \cdot \overset{\text{inward normal}}{\vec{n}} dA = \iiint_D k \text{ div}(\text{grad } u) dV$$

$$= \iiint_D k \Delta u dV$$

Amount of heat energy in D :

$$\iiint_D \sigma \rho u dV$$

σ - specific heat } constants.
 ρ - mass density }

and the rate it is changing is

$$\frac{\partial}{\partial t} \iiint_D \sigma \rho u(x, y, z, t) dx dy dz$$

$$= \iiint_D \sigma \rho \frac{\partial u}{\partial t} dV$$

$$= \iiint_D k \Delta u dV$$

As this is true for all regions, must have

$$k \Delta u = \sigma \rho \frac{\partial u}{\partial t} \Rightarrow \frac{\partial u}{\partial t} = \frac{k}{\sigma \rho} \Delta u.$$

Unit check: Heat energy is in J, so flux is J/s = W

$$k \text{ in } \frac{\text{W}}{\text{mK}}, \quad \rho \text{ in } \frac{\text{g}}{\text{m}^3}, \quad \sigma = \frac{\text{J}}{\text{gK}}.$$