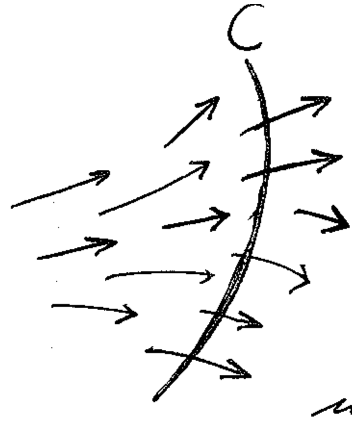
Last time: <u>Flux</u>

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$



Flux = rate water is crossing $\vec{F}$

$= \int_C (\vec{F} \cdot \vec{n})\, ds$

where $\vec{n}$ is a unit normal vector field along C.

[Today, will use the concept of flux to give a different formulation of Green's Thm, one that will explain why it works, and will also generalize to $\mathbb{R}^3$]

<u>Divergence</u>: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field with $\vec{F} = (F_1, F_2)$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$$

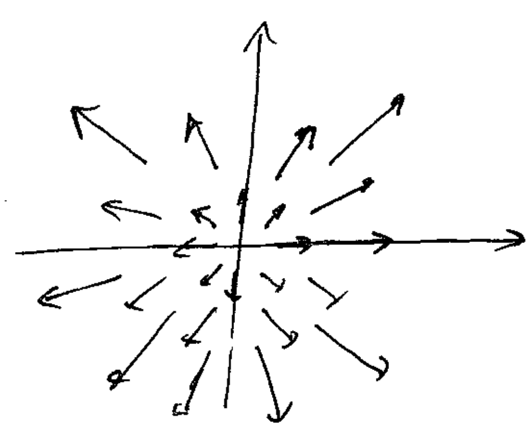
which is [Q.] a function $\mathbb{R}^2 \to \mathbb{R}$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

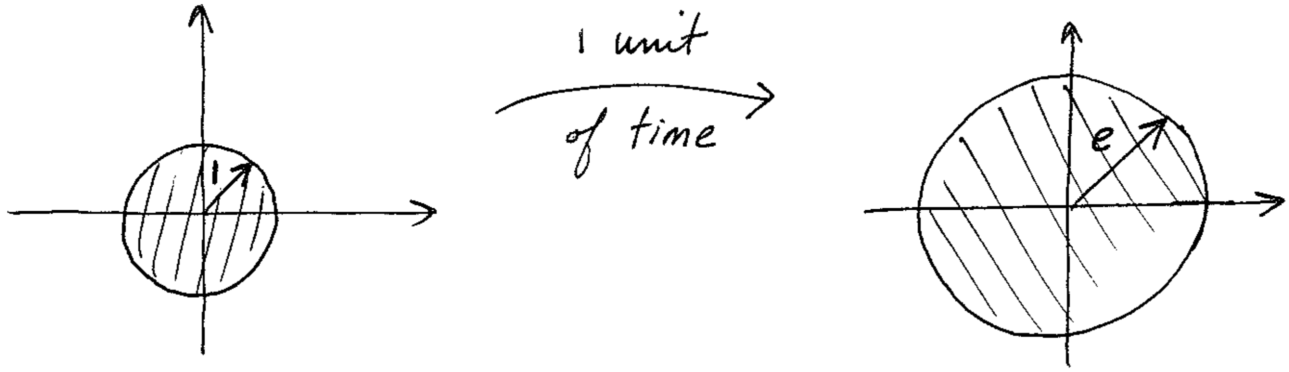
<u>Meaning</u>: [Thinking of $\vec{F}$ as rep. fluid flow:]

$\operatorname{div} \vec{F}$ = rate of expansion of area under the flow.

<u>Ex</u>: $\vec{F} = (x, y)$

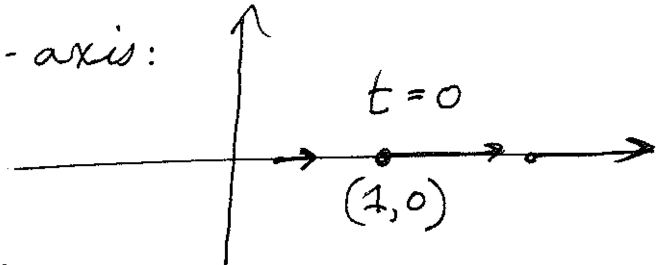


Suppose we dye the water inside the unit circle green.



<u>Reason</u>: Consider the flow on the x-axis:

$$x'(t) = x(t) \Rightarrow x(t) = e^t$$



In general, the follow of (x_0, y_0) is given by $\vec{f}(t) = (x_0 e^t, y_0 e^t)$ since $\vec{f}'(t) = \vec{f}(t) = \vec{F}(\vec{f}(t))$

So

$$\frac{\text{Green Area}(t)}{\text{Green Area @ } t=0} = \frac{2\pi (e^t)^2}{2\pi} = e^{2t}$$

rate of increase in the green area.

Note

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2.$$

match!

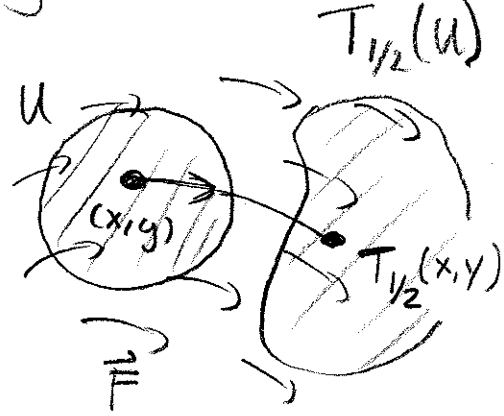
In real life, how could this happen?

ⓐ Fluid becomes more/less dense (compressible)

[When modeling airflow over a wing, usually assume $\operatorname{div} \vec{F} = 0$ if speeds are < Mach 0.3]

ⓑ Fluid is being added somehow.

For any Δt, get a transformation $T_{\Delta t} : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule $T_{\Delta t}(x,y) =$ position of (x,y) after flowing for time Δt.



For small Δt, setting $\vec{F} = (P,Q)$ have

$$T_{\Delta t}(x,y) \approx (x,y) + \Delta t\, \vec{F}(x,y)$$

$$= (x + P(x,y)\Delta t,\ y + Q(x,y)\Delta t)$$

The Jacobian of $T_{\Delta t}$ at (x,y) is about

$$\begin{vmatrix} 1 + P_x(x,y)\Delta t & P_y(x,y)\Delta t \\ Q_x(x,y)\Delta t & 1 + Q_y(x,y)\Delta t \end{vmatrix}$$

$$= 1 + (P_x(x,y) + Q_y(x,y))\Delta t + (\text{stuff})\Delta t^2.$$

So the rate of expansion of area is

$$P_x(x,y) + Q_y(x,y) = \operatorname{div} \vec{F}$$

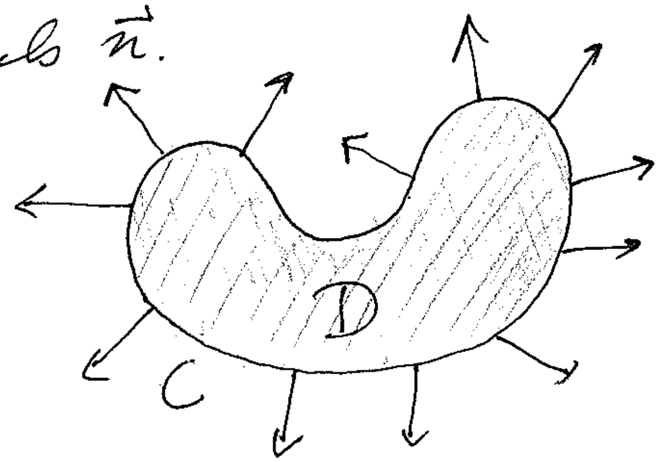
Check units:

$\vec{F}(x,y)$ has units m/s

$\operatorname{div}\vec{F}$ has units $1/s$ and so $1+\Delta t(\operatorname{div}\vec{F})$ is dimensionless.

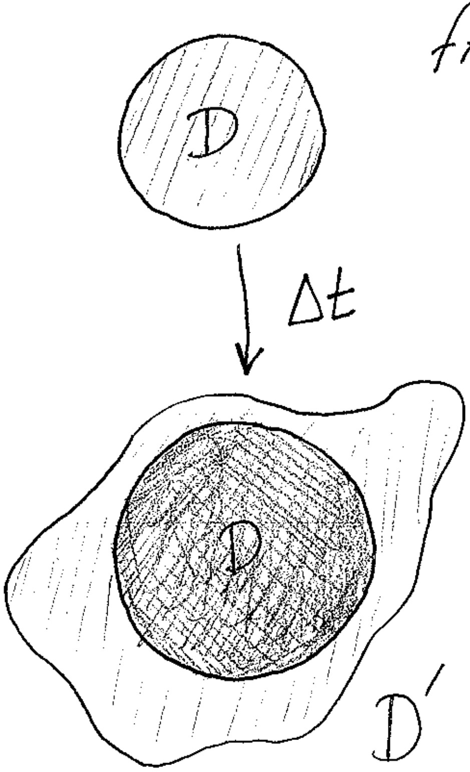
Divergence Thm: D a region in $\mathbb{R}^2$ bounded by C, with outward unit normals $\vec{n}$.

Then



$$\underbrace{\int_C \vec{F}\cdot\vec{n}\,ds}_{\text{Flux}} = \iint_D \operatorname{div}\vec{F}\,dA$$

Reason: After time Δt, the fluid in D now fills the region D'. Now



$$\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1 + \Delta t \cdot r$$

where $r = \underset{\text{of expansion}}{\text{Ave. rate}} = \frac{1}{\text{Area}(D)}\iint_D \operatorname{div}\vec{F}\,dA$

$\Longrightarrow$

$$\text{Area}(D') - \text{Area}(D) \approx \Delta t \iint_D \operatorname{div}\vec{F}\,dA$$

Now as the region D is fixed, the change in area can only be accomplished by fluid crossing C. The amount that crosses C in time Δt is $\approx \Delta t \int_C (\vec{F} \cdot \vec{n})\, ds$. Thus we must have $\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}\, dA$

How this relates to Green's Thm:

If $\vec{F} = (F_1, F_2)$ set $\vec{G} = (-F_2, F_1) = (P, Q)$

Then

$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_C \vec{G} \cdot d\vec{r} \qquad (\star)$$

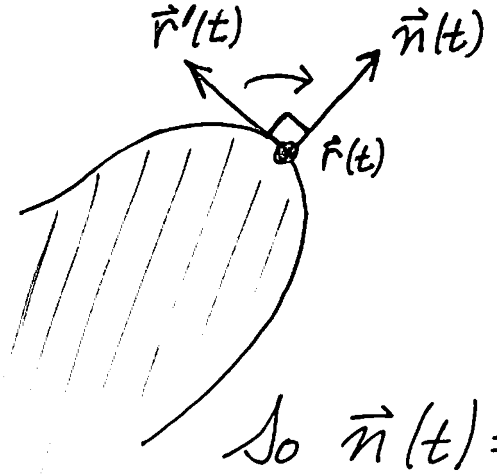
and

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

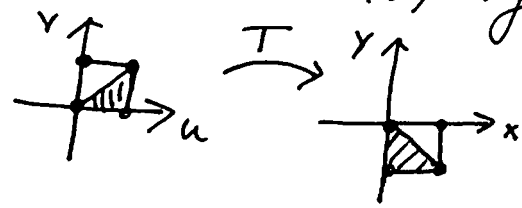
so

$$\iint_D \operatorname{div} \vec{F}\, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Reason for (*): Take a unit-speed param $\vec{r}: [a,b] \to C$.



Get $\vec{n}(t)$ from $\vec{r}'(t)$ by rotating right



$$T(u,v) = (v,-u)$$

So $\vec{n}(t) = (r_2'(t), -r_1'(t))$ and

$$\int_C \vec{F}\cdot\vec{n}\, ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{n}(t)\, dt$$

$$= \int_a^b F_1(\vec{r}(t))\, r_2'(t) - F_2(\vec{r}(t))\, r_1'(t)\, dt$$

$$= \int_a^b G(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$$

$$= \int_C G\cdot d\vec{r}$$

Next time: Flux across a surface