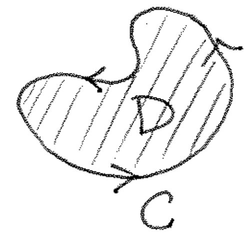
<u>Previously:</u> D region in $\mathbb{R}^2$ bounded by a curve C

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ a vector field with $\vec{F} = (P, Q)$

<u>Green's Thm:</u> $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$



<u>Thm A:</u> $\vec{F}$ on a connected open set D in $\mathbb{R}^2$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C in D.

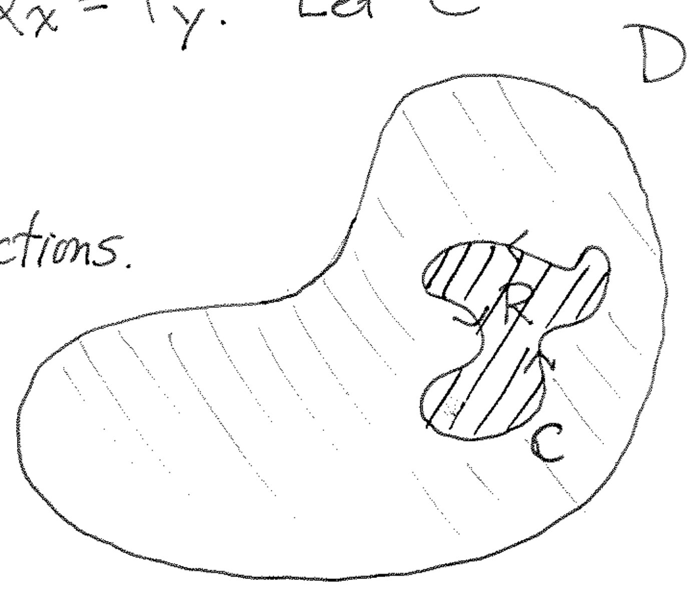
<u>Thm B:</u> If D in $\mathbb{R}^2$ is simply connected, then $\vec{F} = (P, Q)$ is conservative if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

[Earier, explained why Thm A works, and that any conserv. vector field on $\mathbb{R}^2$ satisfies $Q_x = P_y$ ...]

<u>Reason for Thm B:</u> Suppose D is simply connected and $F = (P, Q)$ satisfies $Q_x = P_y$. Let C be a closed curve in D.

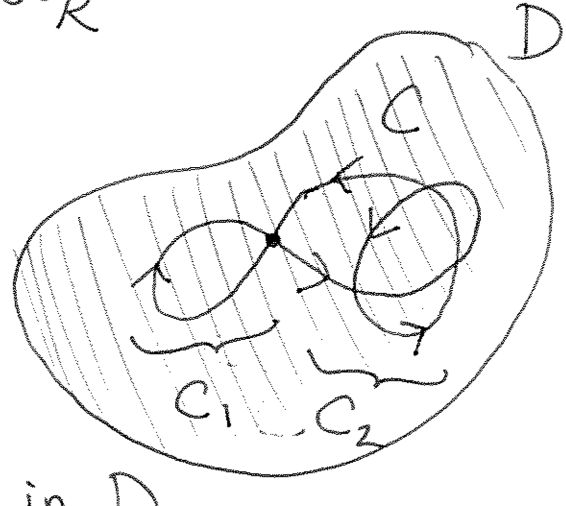
<u>Case 1:</u> C has no self-intersections.

Then C is the boundary of a region R inside D



since D "has no holes". Now, by Green's Thm

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R Q_x - P_y \, dA = \iint_R 0 \, dA = 0$$



<u>Case 2</u>: C has self-intersections:

Looking at the first time C meets itself, write $C = C_1 + C_2$ where each C_i is a closed curve and C_1 bounds a region R in D.

Then $\int_{C_1} \vec{F}\cdot d\vec{r} = 0$ as before, and so $\int_C \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$. Repeating the argument with C_2 eventually shows $\int_C \vec{F}\cdot dr = 0$.

<u>Conclusion</u>: Theorem A applies to show $\vec{F}$ is conservative

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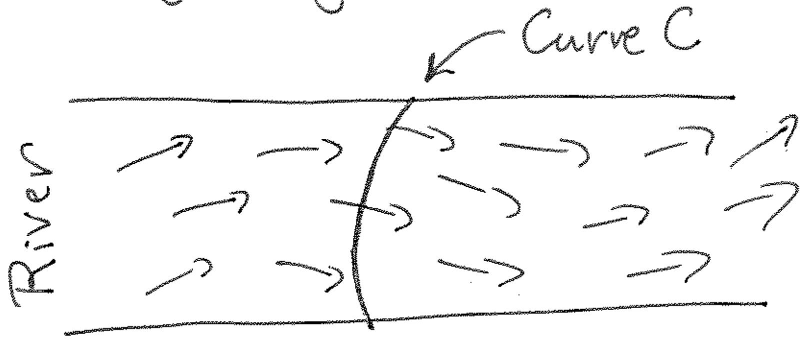
<u>Integrating</u> Vector Fields: Flux in 2D (§16.5)

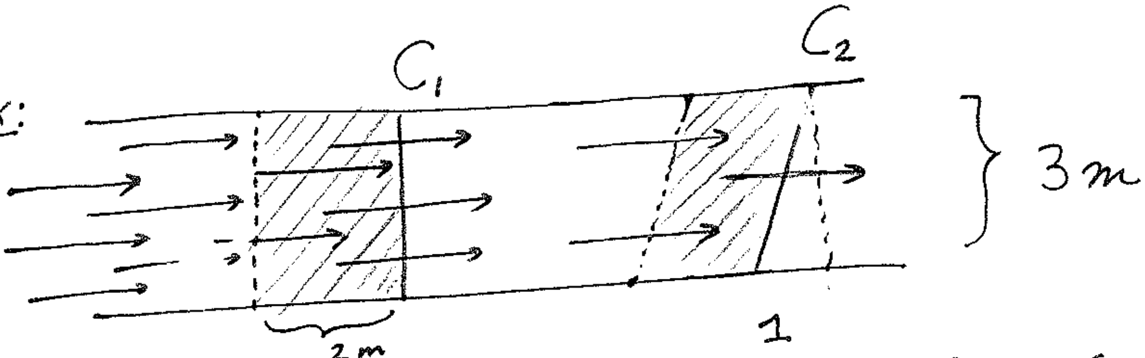


$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

<u>Q</u>: What is the rate that the fluid is crossing C? (= Flux)

Note: Answer will have units (area)/(time) since this is a 2^d- flow.

Ex:



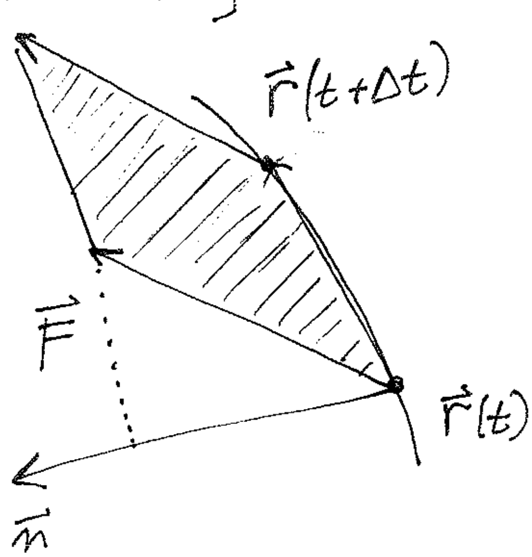
$\vec{F} = (2\ m/s, 0)$

Flux across $C_1 = 6\ m^2/s$

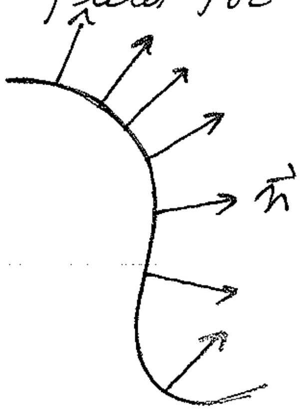
Flux across $C_2 = 6\ m^2/s$

[Q: Should this be more or less than with C_1?]

Closeup of a general curve, where $\vec{F}$ is essentially constant



Let $\vec{n}$ be a unit normal vector field for C.

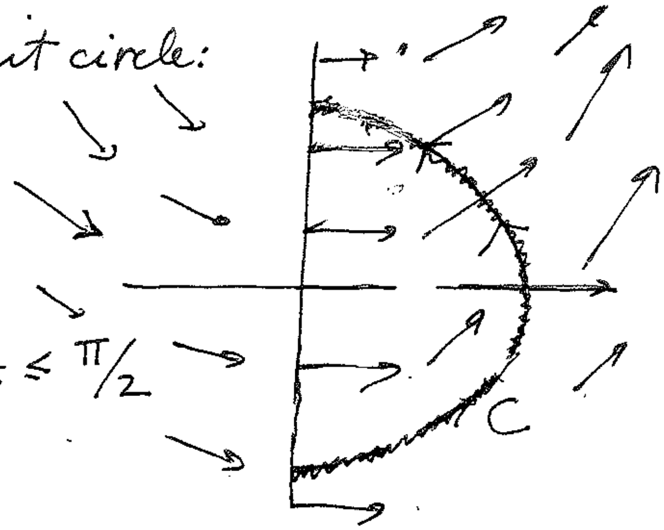


Area of fluid crossing segment.

$\approx (\vec{F}\cdot\vec{n})\,|\vec{r}\,'(t)|\,\Delta t.$

$$\text{Hence Flux} = \int_C \overbrace{(\vec{F}\cdot\vec{n})}^{\text{function along the curve.}}\, ds$$

Ex: $\vec{F} = (1, x)$ and $C = \frac{1}{2}$ unit circle:



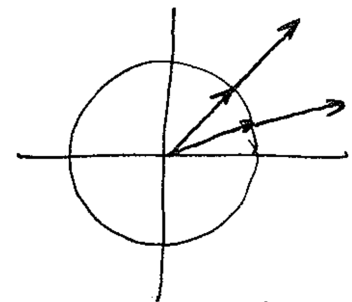
Parameterization:

$$\vec{r}(t) = (\cos t, \sin t) \quad -\pi/2 \le t \le \pi/2$$

$$\vec{n}(t) = (\cos t, \sin t)$$

Special to this situtation!

Check:



$$\vec{n}(t) \cdot \vec{r}\,'(t) = (\cos t, \sin t) \cdot (-\sin t, \cos t)$$
$$= 0.$$

$$|\vec{n}(t)| = 1.$$

Flux:

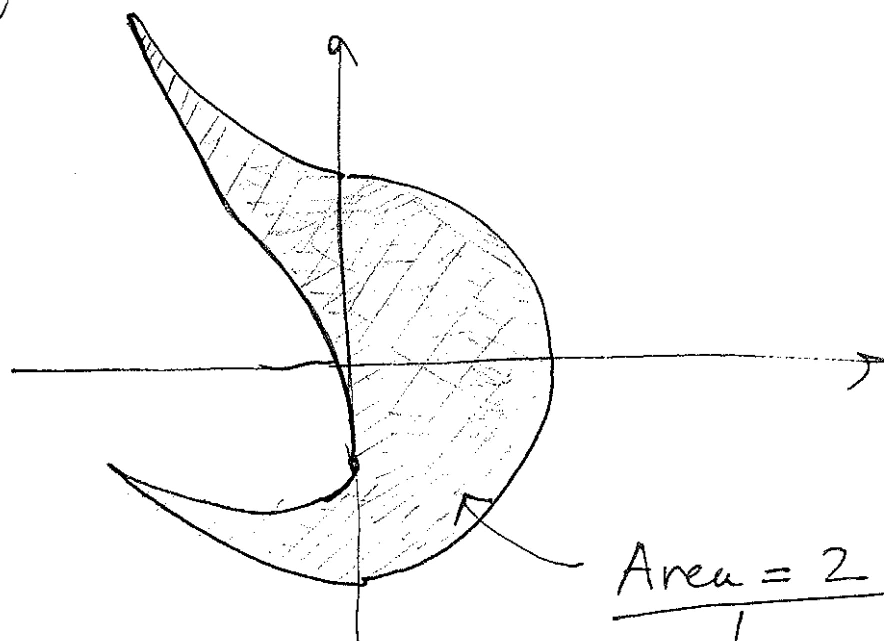
$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \overbrace{\underbrace{|\vec{r}\,'(t)|}_{1}\, dt}^{ds}$$

$$= \int_{-\pi/2}^{\pi/2} (1, \cos t) \cdot (\cos t, \sin t)\; dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos t + \sin t \cos t \; dt = 2$$

What does the region of water look like which flows across C in one unit of time?

A.

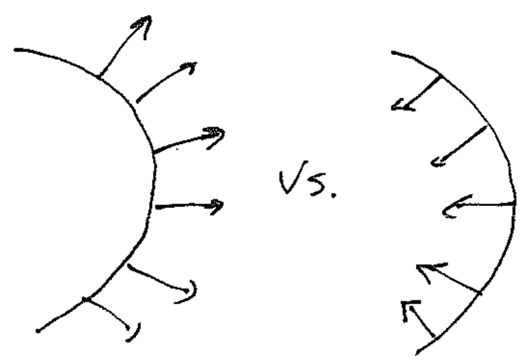


Area = 2

Note to self: Since $\operatorname{div} \vec{F} = 0$.

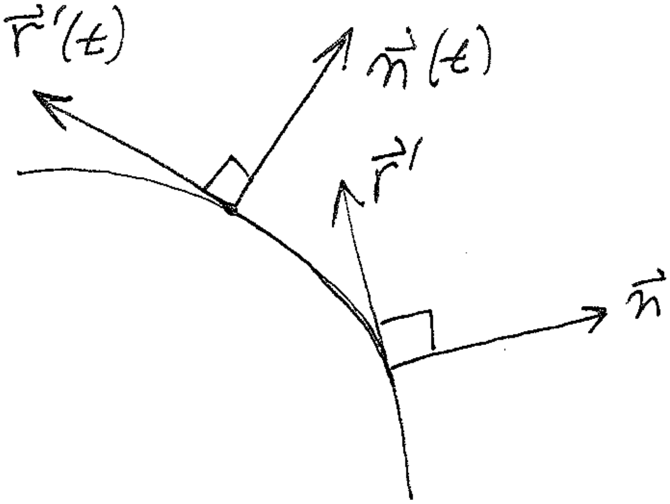
Show animation of this

Note: When you compute the flux, must choose a direction for $\vec{n}$:



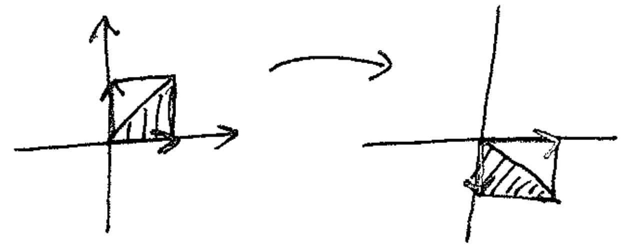
Method for computing $\vec{n}$:

Assume $\vec{r} : [a,b] \longrightarrow C$ has unit speed (e.g. param by arc length).



Relation between:

rotate by 90°. $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$



$T(u,v) = (v, -u)$

So if $\vec r(t) = (r_1(t), r_2(t))$ have

$$\vec n(t) = (+r_2'(t), -r_1'(t))$$

Now suppose $\vec F = (F_1, F_2)$. Then

$$\int_C (\vec F \cdot \vec n)\, ds = \int_a^b (F_1(\vec r(t)), F_2(\vec r(t))) \cdot (r_2'(t), -r_1'(t))\, \overset{\text{unitspeed} \atop = ds}{\widetilde{dt}}$$

$$= \int_a^b \vec F_1(\vec r(t))\, r_2'(t) - \vec F_2(\vec r(t))\, \vec r_1'(t)\, dt$$

$$= \int_C \vec G \cdot d\vec r \quad \text{where } \vec G = (-F_2, F_1)$$

$$= \iint_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA$$

What does this measure?

Green's Thm, assuming $C = \partial D$