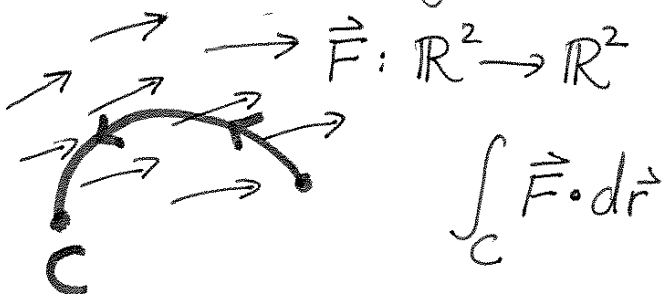


Lecture 30: Green's Theorem (§16.4)

Previously: Some of the new kinds of integrals in this course:

1) Vector fields

along curves:



2) Functions on 2D

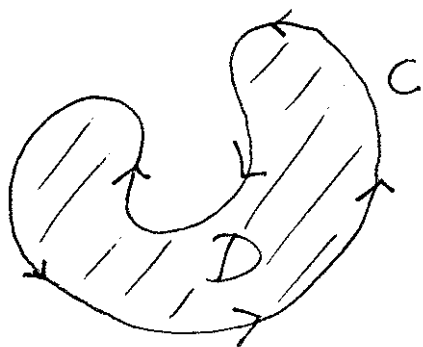
regions:

$$\iint_D f \, dA$$



[Onward to new versions of the Fund. Thm of Calc.]

C a closed curve bounding a region D in \mathbb{R}^2 ,



oriented so D is to the left
as you go around C .

$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ a vector field

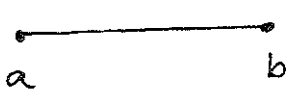
given by $\vec{F}(x, y) = (P(x, y), Q(x, y))$

Green's Thm:

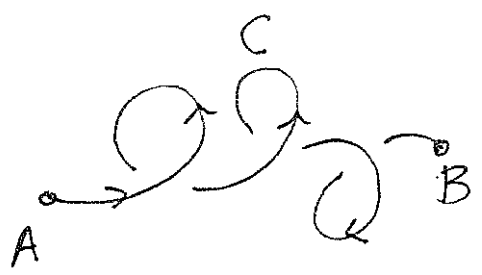
$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

[Seems mysterious: how can an integral over
the whole region depend only \vec{F} along the
boundary curve C .]

Compare:

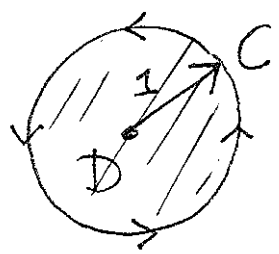


$$f(b) - f(a) = \int_a^b f'(x) dx$$

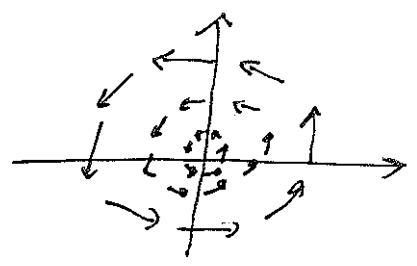


$$f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$$

Ex:



$$\vec{F} = \frac{1}{2}(-y, x)$$



$$\vec{r}(t) = (\cos t, \sin t) \quad 0 \leq t \leq 2\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

$$= \int_0^{2\pi} \frac{1}{2}(-\sin t, \cos t) \cdot (-\sin t, \cos t) dt$$

$$= \frac{1}{2} \int_0^{2\pi} (-\sin t)^2 + (\cos t)^2 dt = \frac{1}{2} \int_0^{2\pi} 1 dt = \boxed{\pi}$$

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \iint_D \frac{1}{2}(1 - (-1)) dA = \iint_D 1 dA$$

$$= \text{Area}(D) = \boxed{\pi}$$

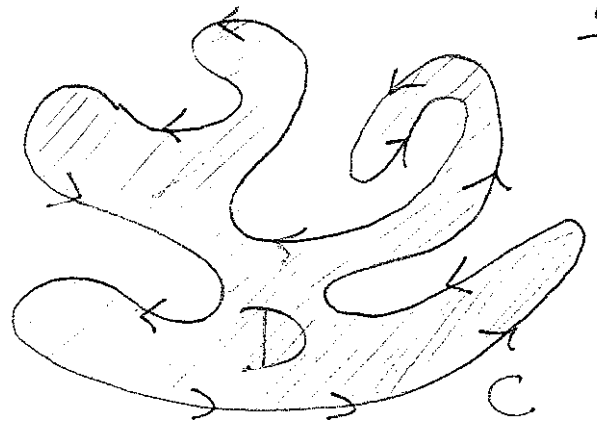
Same answer both times!

Special Case: D any region

Take $\vec{F} = \frac{1}{2}(-y, x)$

Then

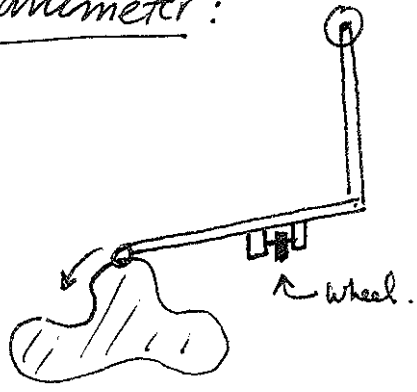
$$\text{Area}(D) = \int_C \vec{F} \cdot d\vec{r}$$



Planimeter:

Fixed to table

The planimeter is a still-manufactured mechanical device that uses this to measure areas

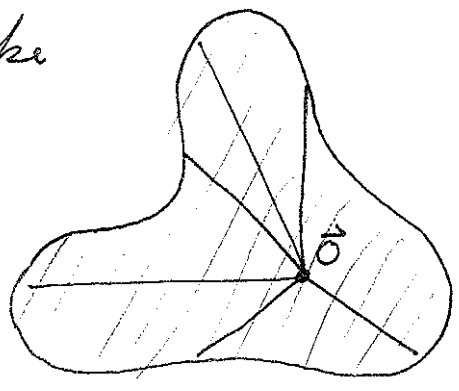


Perfected in 1854... see web link for details.

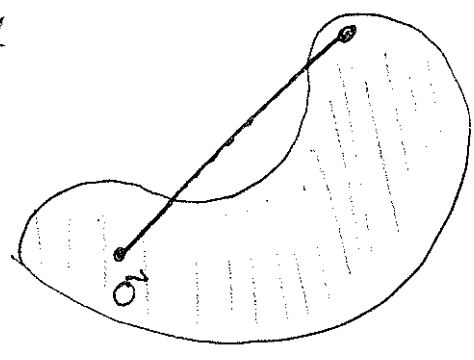
Why Green's Theorem works in this special case:

Suppose \vec{o} is in D , which is "star shaped"

like



not

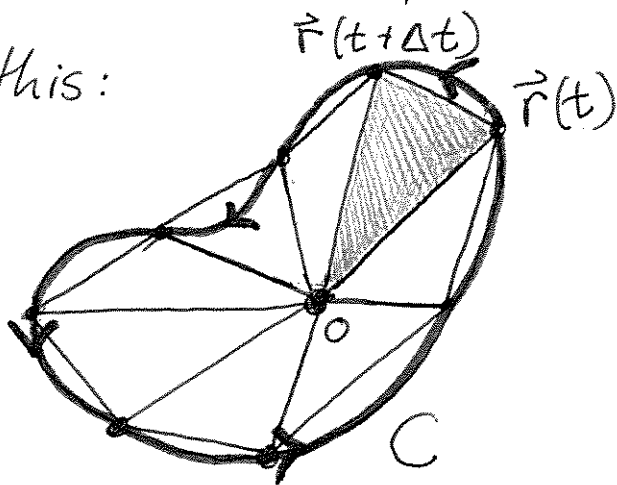
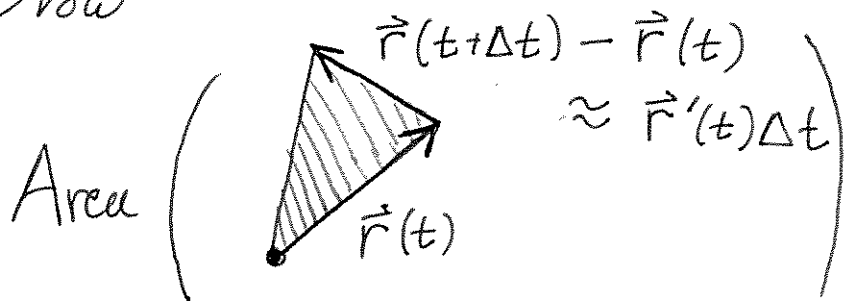


[i.e. can see entire room D from \vec{o} . Is this room star shaped?]

Suppose $\vec{r}: [a, b] \rightarrow \mathbb{R}^2$ is a parameterization of C .

Use it to estimate $\text{Area}(D)$ like this:

Now



$$\approx \frac{1}{2} \text{Area} \left(\begin{array}{l} \vec{r}'(t)\Delta t = (r_1'(t)\Delta t, r_2'(t)\Delta t) \\ \vec{r}(t) = (r_1(t), r_2(t)) \end{array} \right)$$

$$= \frac{1}{2} \begin{vmatrix} r_1(t) & r_1'(t)\Delta t \\ r_2(t) & r_2'(t)\Delta t \end{vmatrix} = \frac{1}{2} (r_1(t)r_2'(t) - r_2(t)r_1'(t))\Delta t$$

Adding up the areas of such triangles gives a Riemann sum, and so taking $\Delta t \rightarrow 0$ we

learn:

$$\text{Area}(D) = \int_a^b \frac{1}{2} (r_1(t)r_2'(t) - r_2(t)r_1'(t)) dt$$

[Pause for questions]

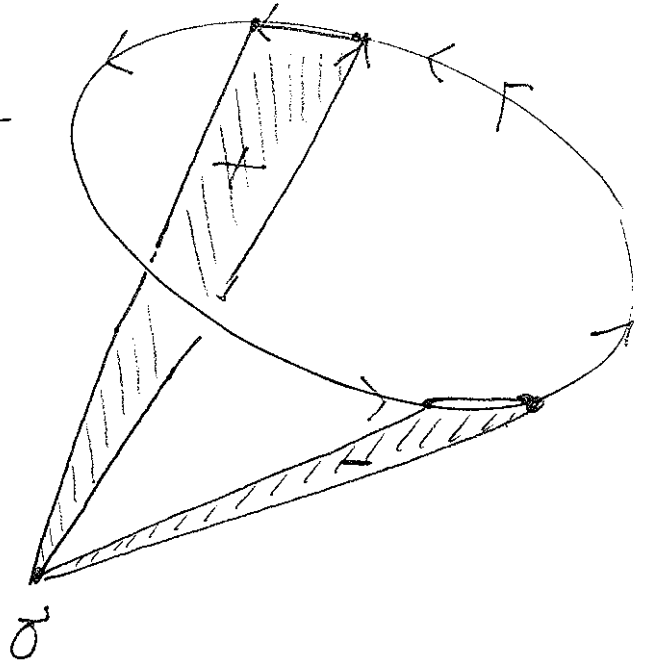
$$= \int_a^b \frac{1}{2} (-r_2(t), r_1(t)) \cdot (r_1'(t), r_2'(t)) dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_C \vec{F} \cdot d\vec{r}$$

What if σ is not in D ?

Then the Riemann sum includes both positive and negative contributions.

The latter cancel out the portion of the pos. contribs that are too large, and so again



$$\text{Area}(D) = \int_C \vec{F} \cdot d\vec{r}$$

• Also works when D has more than one boundary component:

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \sum_{\substack{\text{boundary} \\ \text{curves } C_i}} \int_{C_i} \vec{F} \cdot d\vec{r}$$



$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

as long as you orient the C_i so that D is always to the left.