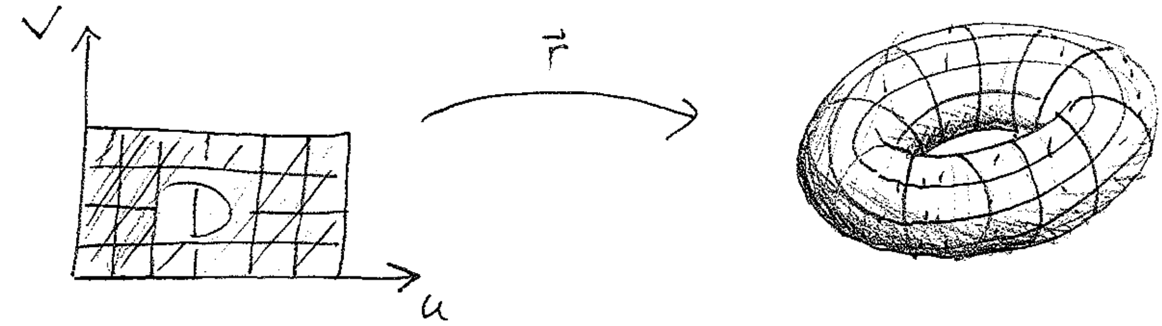
Lecture 29: Area and integration on surfaces (§16.6 and §16.7)

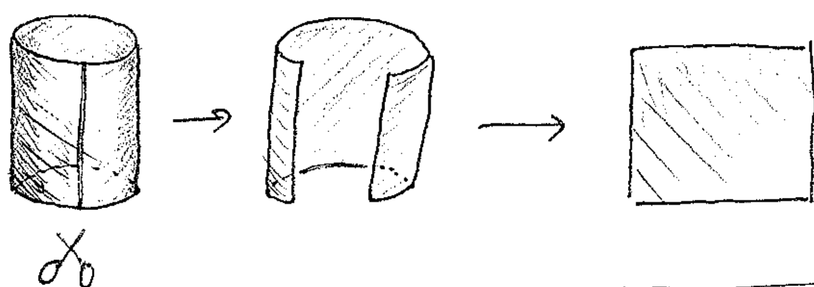
Last time: Parameterizing surfaces



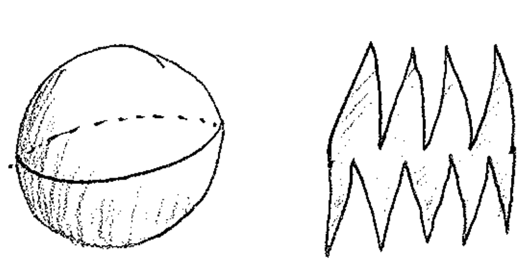
$$\vec{r} : (D \text{ in } \mathbb{R}^2) \longrightarrow (S \text{ in } \mathbb{R}^3)$$

Q: How do we compute the area of S?

Easy:

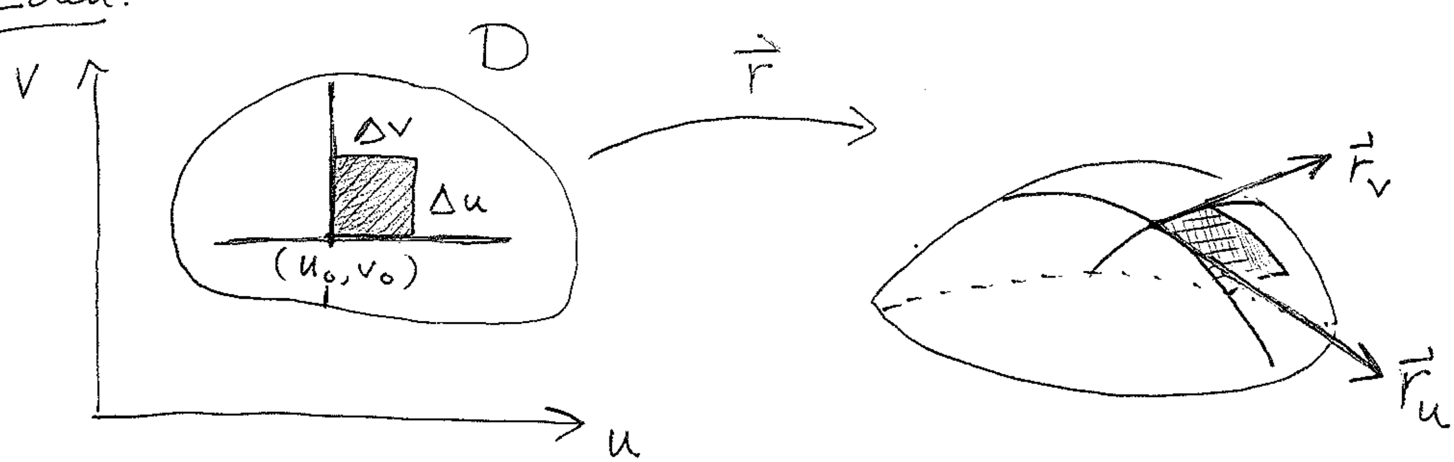


Hard:

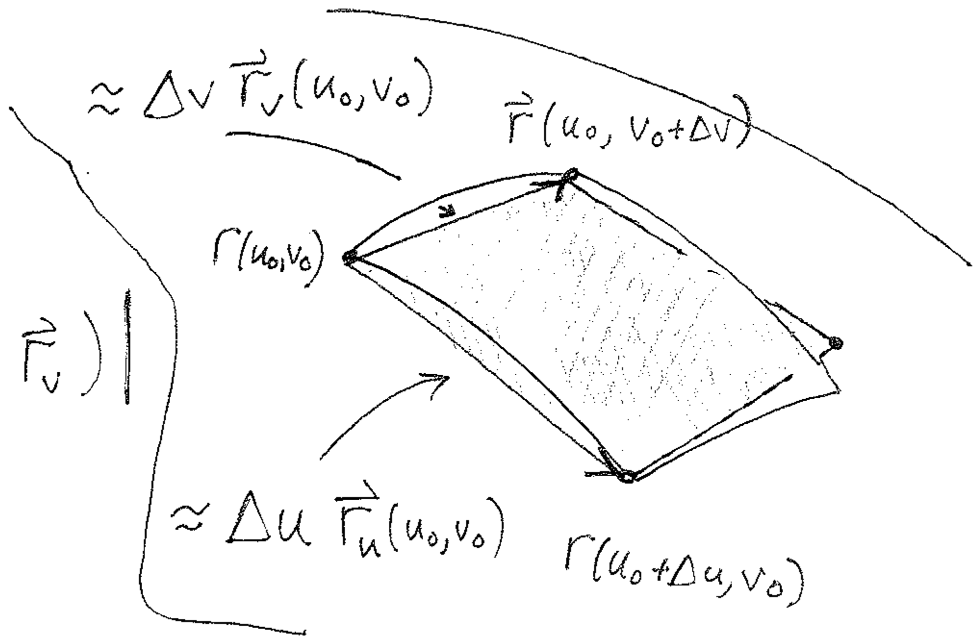


Point: Really, the same idea as change of coordinates, basic line ints, etc...

Idea:



We approximate the area of region at right using a parallelogram:



So the area is

$$\approx \left| (\Delta u\, \vec{r}_u) \times (\Delta v\, \vec{r}_v) \right|$$

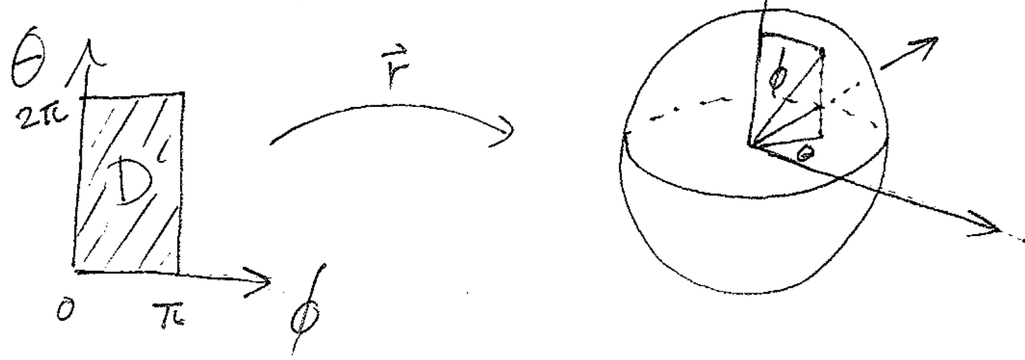
$$= |\vec{r}_u \times \vec{r}_v| \,\Delta u \,\Delta v$$

So,

$$\text{Area}(S) \approx \sum_{\substack{\text{small}\\ \text{rects}}} |\vec{r}_u \times \vec{r}_v| \,\Delta u\, \Delta v.$$

Let $\Delta u, \Delta v \to 0$, then $\text{Area}(S) = \iint_D |\vec{r}_u \times \vec{r}_v| \, du\, dv$

<u>Ex:</u> Unit sphere



$$\vec{r}(\phi, \theta) = (\sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi)$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

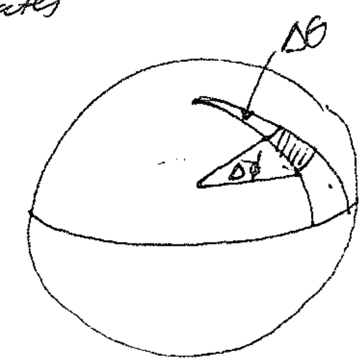
$$= \left(+\sin^2\phi\cos\theta,\ -\sin^2\phi\sin\theta,\ \sin\phi\cos\phi\left(\cos^2\theta+\sin^2\theta\right) \right)$$

$$|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\sin^4\phi + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi} = \sin\phi$$

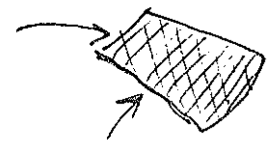
as $0 \le \phi \le \pi$.

Geometrically, we saw this $\sin\phi$ before when we were discussing spherical coordinates

Aside



len $= \sin\phi\,\Delta\theta$



len $= \Delta\phi$

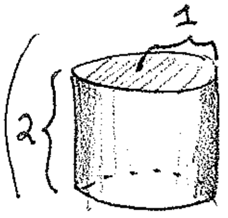
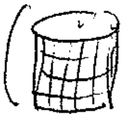
Area $\approx \sin\phi\,\Delta\theta\Delta\phi$

$$\text{Area}\left(\bigcirc\right) = \iint_D \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \underbrace{-\cos\phi\Big|_{\phi=0}^{\phi=\pi}}_{= -(-1)-(-1) = 2} d\theta$$

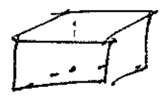
$$= \int_0^{2\pi} 2\, d\theta = \boxed{4\pi}$$

Note: Area() = 2 Area() + Area()

$$= 2\pi + 4\pi = 6\pi.$$

Hence

Area() / Area() $= \dfrac{3}{2} =$ Vol() / Vol()

Isn't typical, e.g. compare a cube  with a sphere.

With curves, finding 

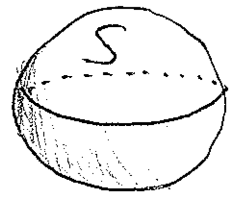
$$\text{Length} = \int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt \quad \text{where } \vec{r}:[a,b] \to C.$$

lead to integrating $f: \mathbb{R}^3 \to \mathbb{R}$ along C via:

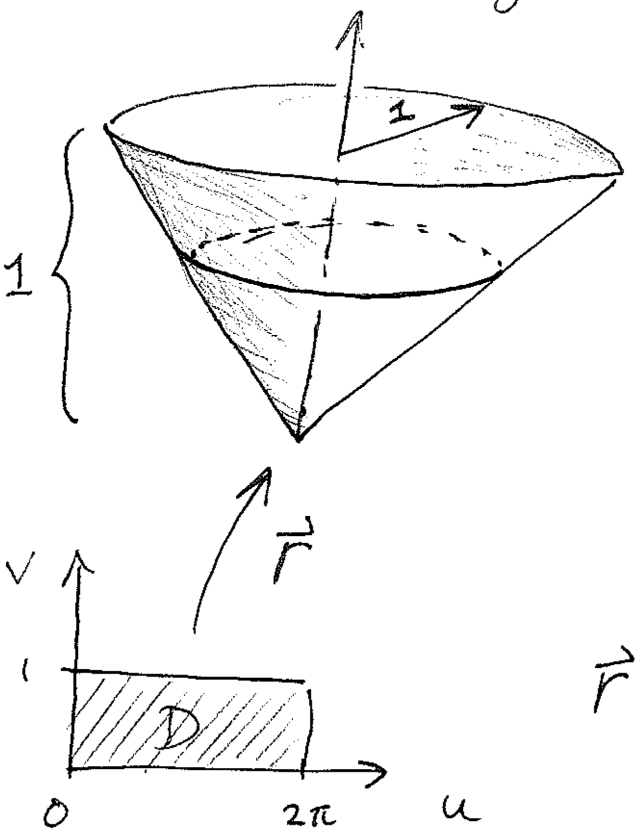
$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

Similarly, if $f: \mathbb{R}^3 \to \mathbb{R}$ and S is a surface,

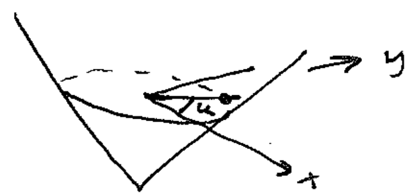
$$\iint_S f\,dA = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

where $\vec{r}: D \to S$ is a parameterization. 

Ex: Find the average of $f(x,y,z) = xy + z$ on the cone S given by $x^2+y^2=z^2$ with $0 \le z \le 1$.



Parameters: height v, angle u



$$\vec{r}(u,v) = (v\cos u,\ v\sin u,\ v)$$

Area: $\vec{r}_u = (-v\sin u,\ v\cos u,\ 0)$

$\vec{r}_v = (\cos u,\ \sin u,\ 1)$

$$|\vec{r}_u \times \vec{r}_v| = |(v\cos u, v\sin u, -v)| = \sqrt{2}\, v$$

So

$$\text{Area} = \iint_S 1\, dA = \iint_D \sqrt{2}\, v\, du\, dv = \int_0^1 \int_0^{2\pi} \sqrt{2} v\, du\, dv$$

$$= \int_0^1 2\sqrt{2}\pi\, v\, dv = \sqrt{2}\pi$$

$$\text{Average} = \frac{1}{\text{Area}} \iint_S xy+z \, dA$$

$$= \frac{1}{\sqrt{2}\pi} \int_0^1 \int_0^{2\pi} (v^2 \sin u \cos u + v)(\sqrt{2}\, v) \, du \, dv$$

$$= \frac{1}{\pi} \int_0^1 \left(\int_0^{2\pi} v^3 \sin u \cos u + v^2 \, du \right) dv$$

$$= \frac{1}{\pi} \int_0^1 \left(\frac{1}{2} v^3 \sin^2 u + v^2 u \Big|_{u=0}^{u=2\pi} \right) dv$$

$$= \frac{1}{\pi} \int_0^1 2\pi v^2 \, dv = 2 \int_0^1 v^2 \, dv = 2 \frac{v^3}{3} \Big|_{v=0}^{1} = \frac{2}{3}.$$