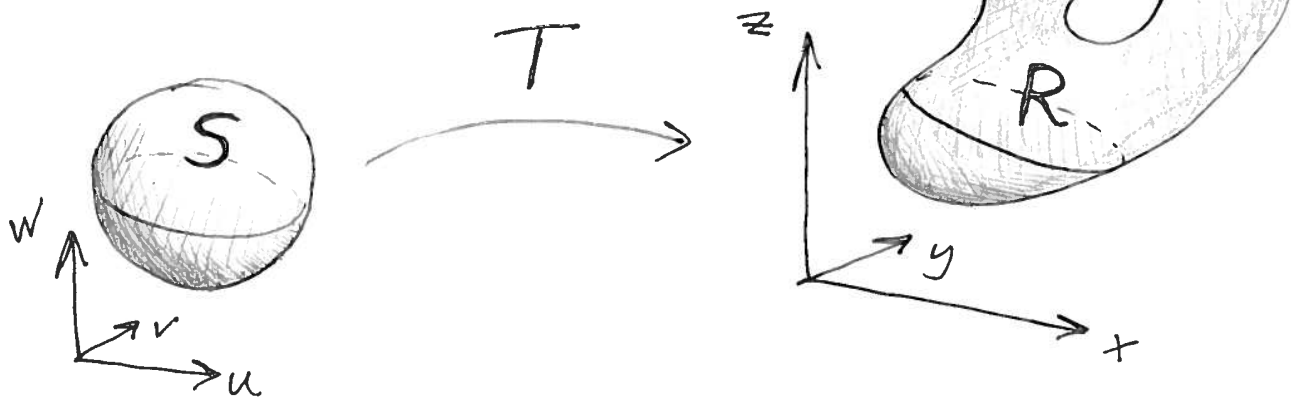


# Lecture 28: Surfaces in $\mathbb{R}^3$ (§16.6)

In general, for  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$

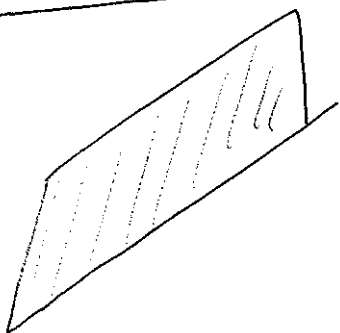


$$\iiint_S f(T(u,v,w)) |\det J| du dv dw = \iiint_R f(x,y,z) dV$$

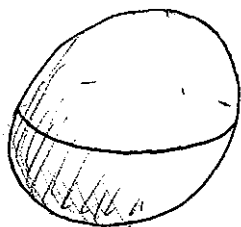
Where

$$J = \begin{pmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{pmatrix} \text{ if } T = (T_1, T_2, T_3).$$

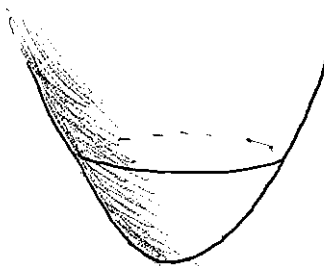
Surfaces in  $\mathbb{R}^3$ : [Things that look locally like  $\mathbb{R}^2$ ]



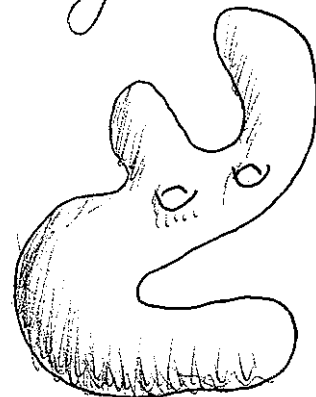
Plane



Sphere



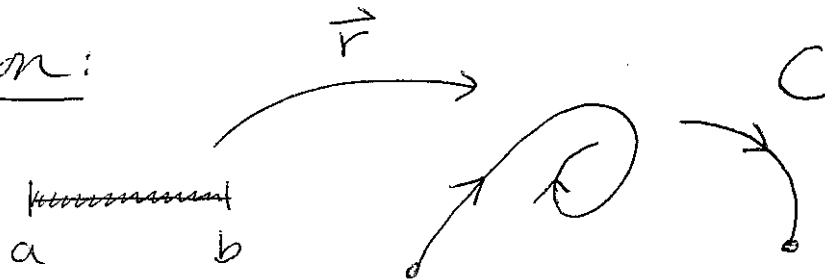
Graph of  
 $f: \mathbb{R}^2 \rightarrow \mathbb{R}$



?

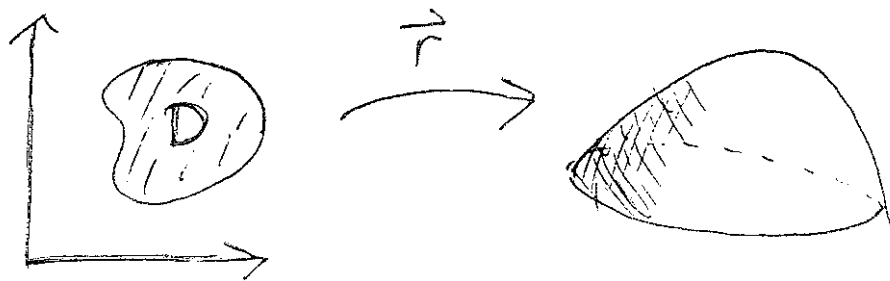
# Parameterization:

Curves:



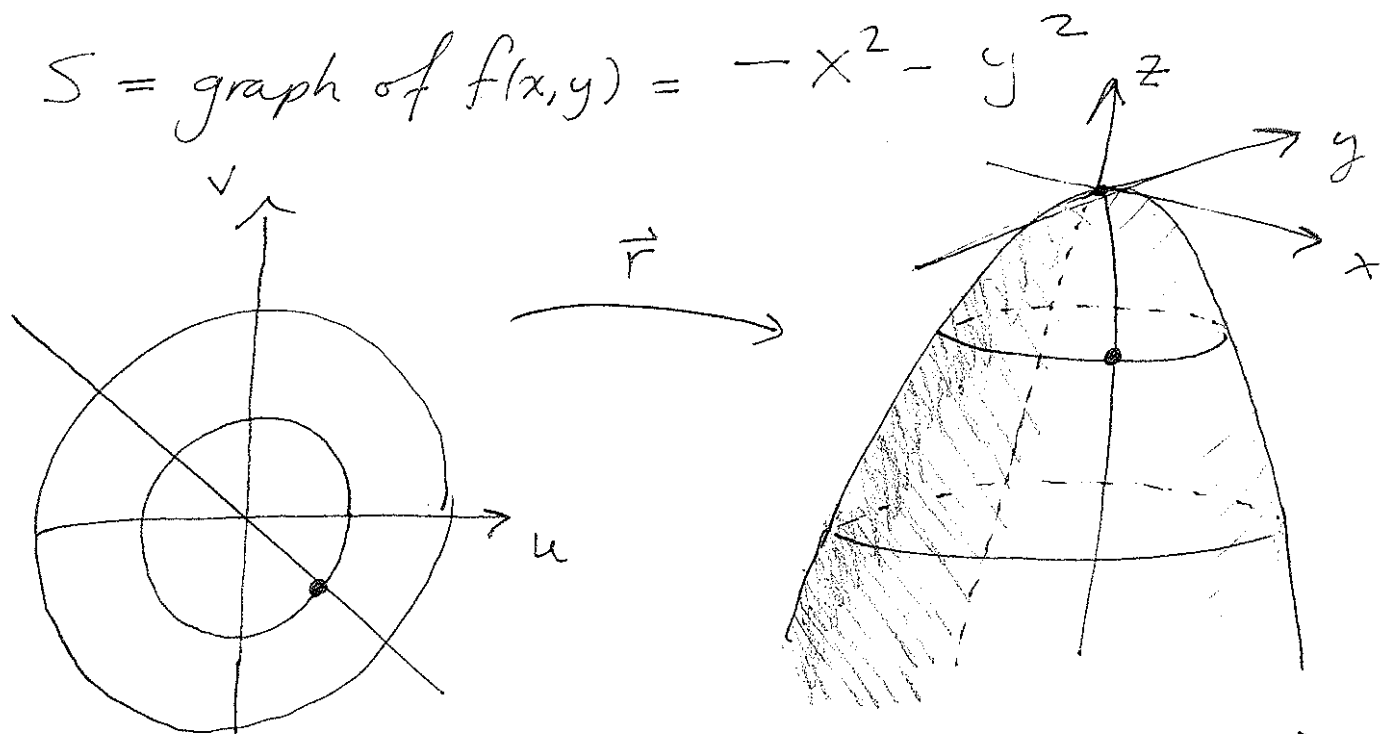
$$\vec{r}: [a, b] \longrightarrow \mathbb{R}^3$$

Surface:



$$\vec{r}: D \longrightarrow \mathbb{R}^3$$

Ex:  $S = \text{graph of } f(x, y) = -x^2 - y^2$

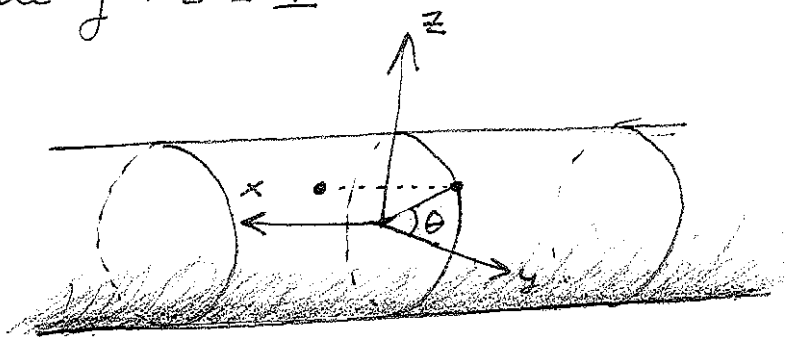


$$\vec{r}: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \text{ given by } \vec{r}(u, v) = (u, v, -u^2 - v^2)$$

Note:  $\vec{r}(u, v) = (u, v, f(u, v))$  works to parameterize the graph of any  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ .

Ex:  $S = \text{cylinder where } y^2 + z^2 = 1$

To specify a point, need to give:



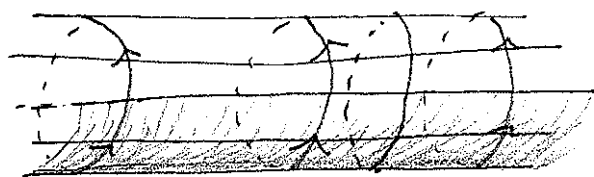
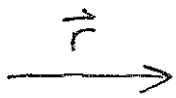
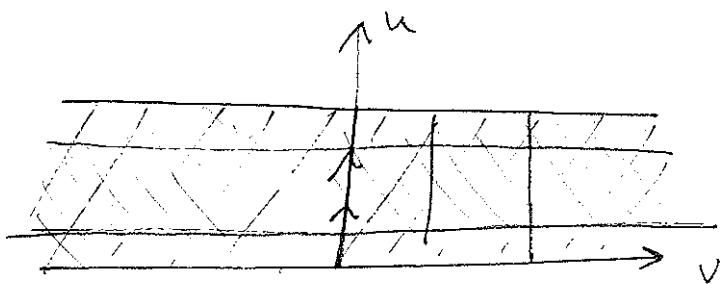
x coord  $\rightarrow u$

$\theta$  angle  $\rightarrow v$

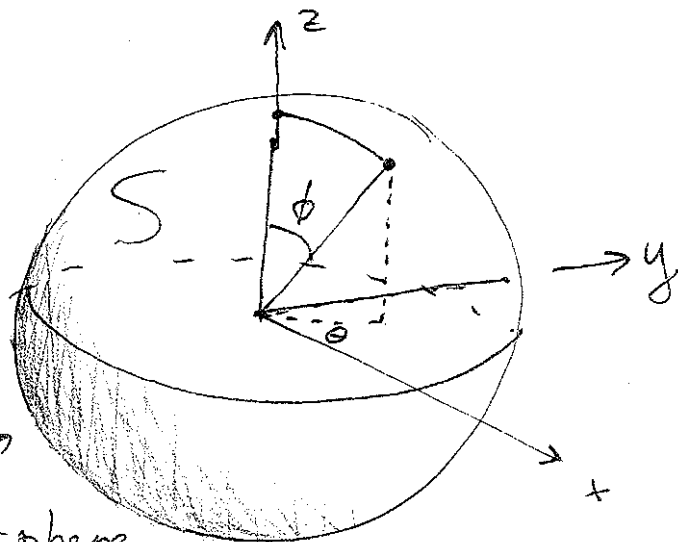
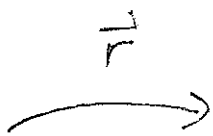
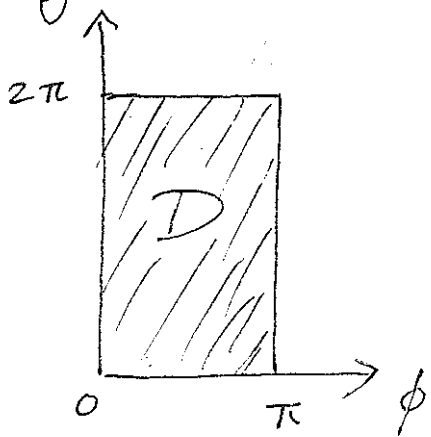
$$\vec{r}: D \rightarrow \mathbb{R}^3$$

$$D = \{ 0 \leq v \leq 2\pi \}$$

$$\vec{r}(u, v) = (u, \cos v, \sin v)$$



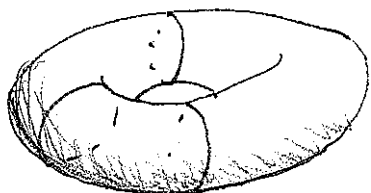
Ex:  $\theta$



unit sphere

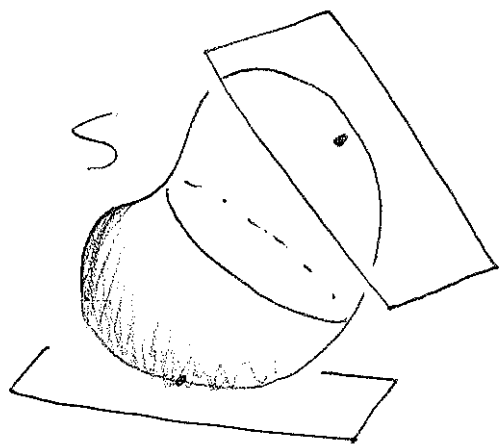
$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

Many more examples:



...

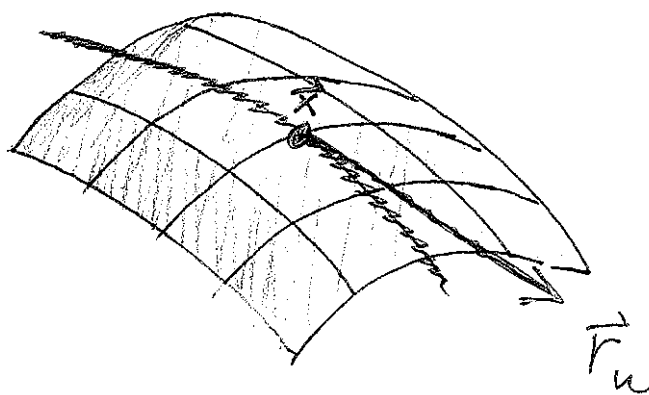
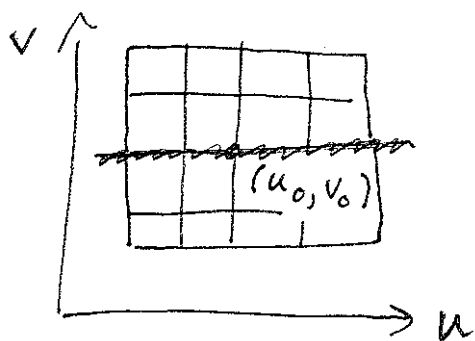
Tangent planes: [Encountered before, for graphs and level sets...]



Locally approximate  $S$ .

[Compare  ~~$\mathbb{R}^3$~~ .]

Consider a param  $\vec{r}: D \rightarrow \mathbb{R}^3$



Consider the curve

$$\vec{f}(t) = \vec{r}(u_0 + t, v_0)$$

which passes through  $\vec{x}$  at time  $t=0$ . Its tangent vector at  $t=0$  is

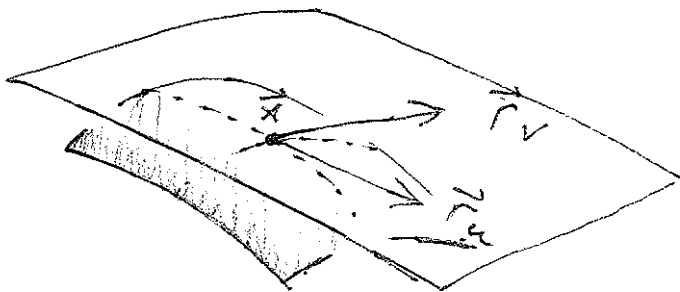
$$\begin{aligned} \vec{f}'(0) &= \left. \frac{d}{dt} (r_1(u_0+t, v_0), r_2(u_0+t, v_0), r_3(u_0+t, v_0)) \right|_{t=0} \\ &= \left( \frac{\partial r_1}{\partial u}(u_0, v_0), \frac{\partial r_2}{\partial u}(u_0, v_0), \frac{\partial r_3}{\partial u}(u_0, v_0) \right) \\ &= \vec{r}_u(u_0, v_0) \end{aligned}$$

where  $\vec{r}(u, v) = (r_1(u, v), r_2(u, v), r_3(u, v))$

Similarly, we have  $\vec{r}_v(u_0, v_0)$

Together,  $\vec{r}_u$  and  $\vec{r}_v$   
span the tangent plane

to  $S$  at  $\vec{x}$ .



Ex: Find the tangent plane to the unit sphere  
at  $(1/\sqrt{2}, 0, 1/\sqrt{2})$ . In terms of

$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

this is  $\vec{r}(\pi/4, 0)$ .

$$\vec{r}_\phi = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi)$$

$$\vec{r}_\phi(\pi/4, 0) = (1/\sqrt{2}, 0, -1/\sqrt{2})$$

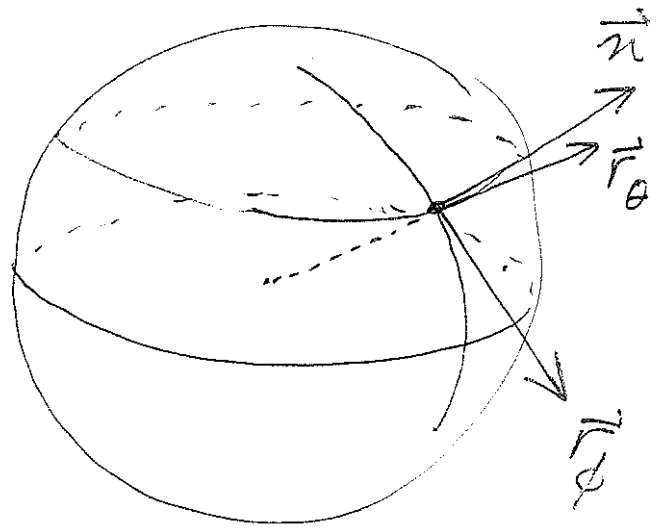
$$\vec{r}_\theta = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)$$

$$\vec{r}_\theta(\pi/4, 0) = (0, 1/\sqrt{2}, 0)$$

So a normal vector is

$$\vec{n} = \vec{r}_\phi \times \vec{r}_\theta$$

$$= \left( \frac{1}{2}, 0, \frac{1}{2} \right)$$



which points straight out from the sphere,  
as we expect.