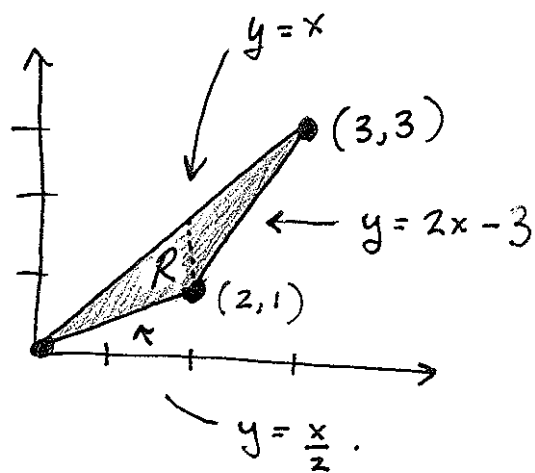


$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad T(r, \theta) = (r \cos \theta, r \sin \theta)$$

$$\iint_R f(x, y) dA = \iint_S f(T(r, \theta)) r dr d\theta$$

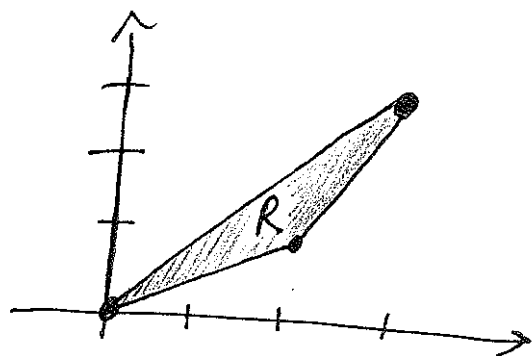
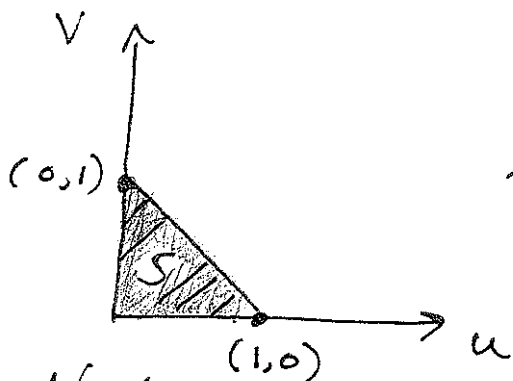
[Next two lectures: general change of coordinates...]

Suppose we want to integrate many functions over the region shown at right. For each, would need two integrals to do so:



$$\iint_R f(x, y) dA = \int_0^2 \int_{\frac{x}{2}}^x f(x, y) dy dx + \int_2^3 \int_{2x-3}^x f(x, y) dy dx.$$

Goal: Do a change of coordinates so that can use just one integral. Sing praises thereof.



Need $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $T(S) = R$.

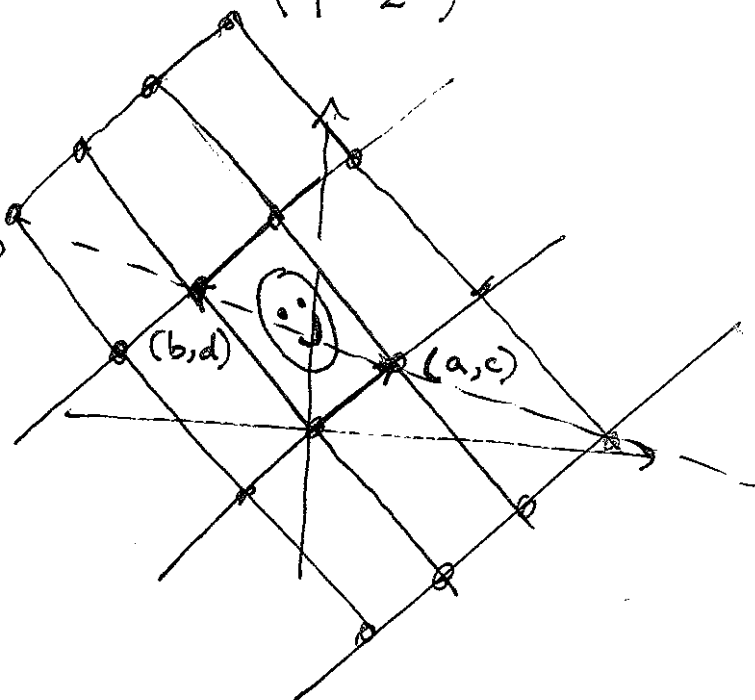
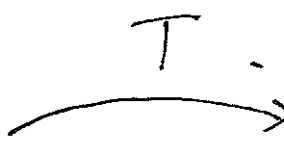
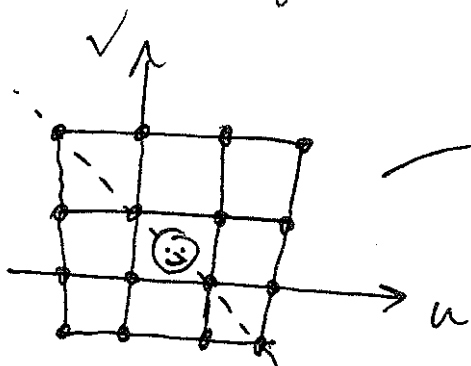
Simplist kind of T : Linear Transformations.

$$T_A(u, v) = (au + bv, cu + dv) \text{ for some } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Ex: $T(u, v) = (u - 2v, u + 2v)$

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$$

from Thursday's worksheet.



Key properties:

(a) $T(\text{Line}) = \text{Line}$

(b) $T(0,0) = (0,0)$

(c) T det. by $T(1,0) = (a, c)$

and $T(0,1) = (b, d)$

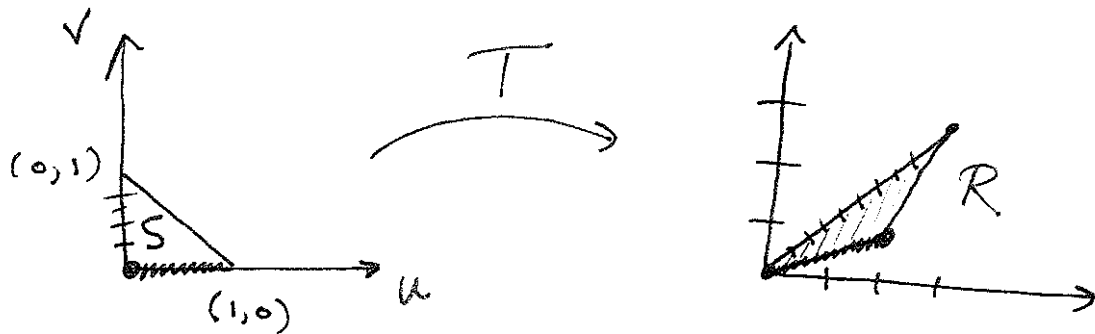
(d) \vec{w}_1, \vec{w}_2 in \mathbb{R}^2 , s, t in \mathbb{R}

$$T(s\vec{w}_1 + t\vec{w}_2) =$$

$$sT(\vec{w}_1) + tT(\vec{w}_2)$$

If we want

$$T(S) = R$$

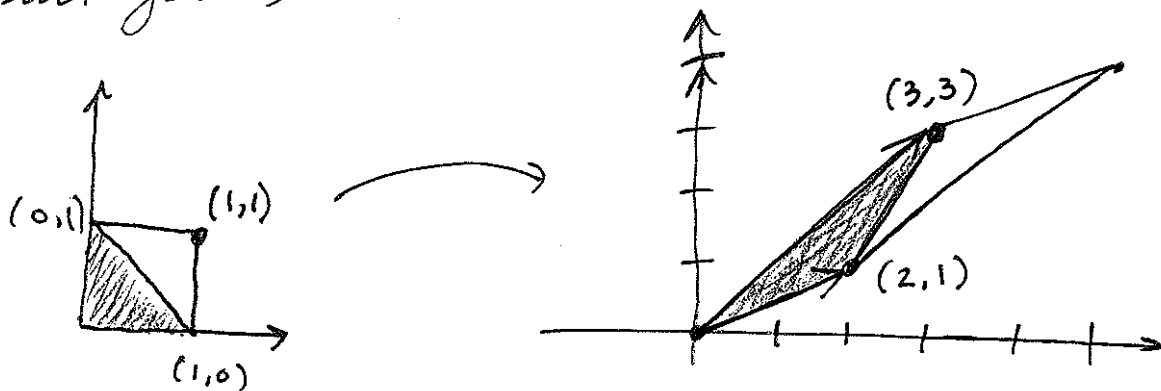


then $T(1,0) = (2,1)$ and $T(0,1) = (3,3)$.

Hence $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $T(u,v) = (2u+3v, u+3v)$

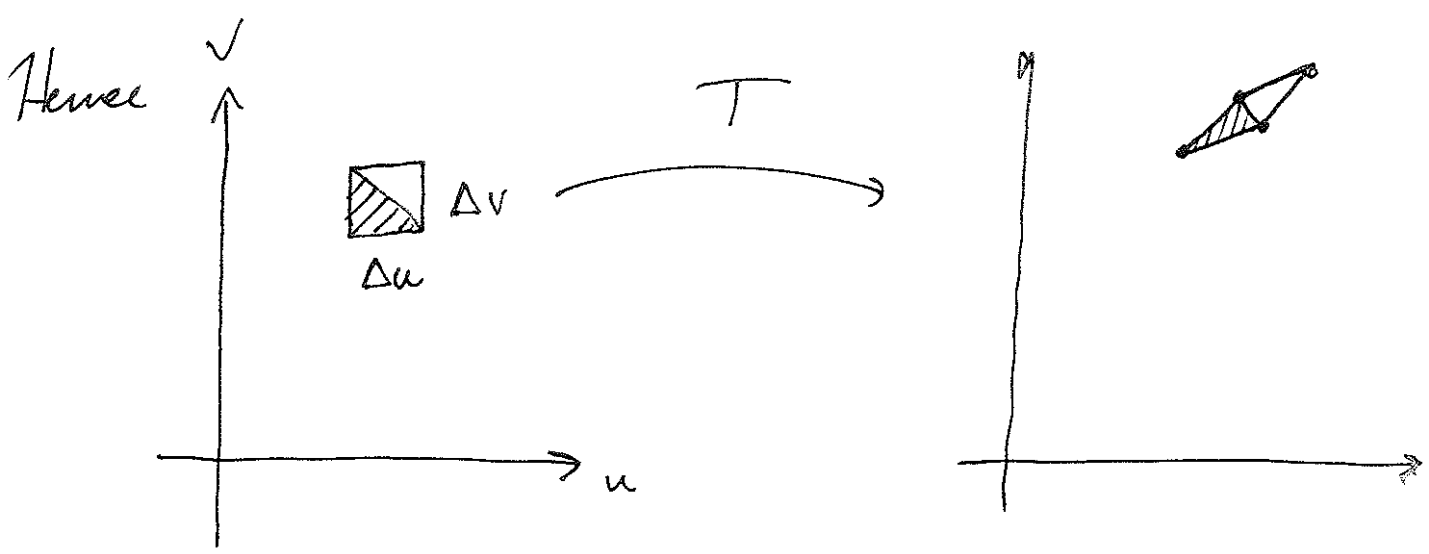
if T is linear, and that will work since such send lines to lines.

To integrate, need to understand how T distorts area.



$$\text{Area}(T(\square)) = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3 \quad T(1,1) = (5,4)$$

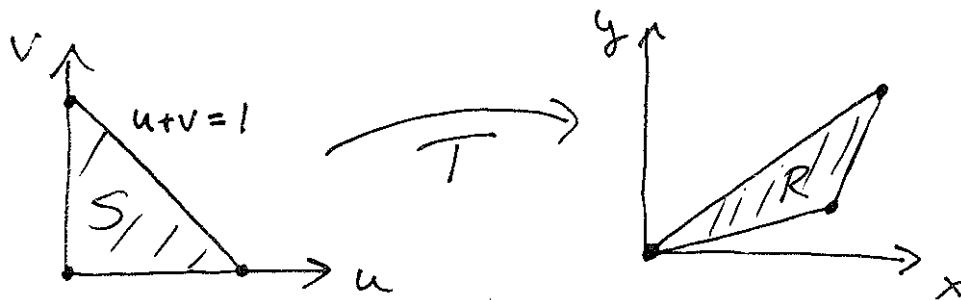
Now a linear transform distorts area uniformly
(think about problem 1 on the worksheet)



$$\text{Area}(\square) = \Delta u \Delta v \longrightarrow \text{Area}(\text{parallelogram}) = 3 \Delta u \Delta v$$

$$\underline{S_0}: dA = 3 du dv$$

$$\underline{\text{Ex}}: \iint_R x - y dA = \iint_S (2u + 3v) - (u + 3v) 3 du dv$$



$$T(u, v) = (2u + 3v, u + 3v) = (x, y)$$

$$= \int_0^1 \int_0^{1-u} u \cdot 3 dv du = \int_0^1 3u(1-u) du$$

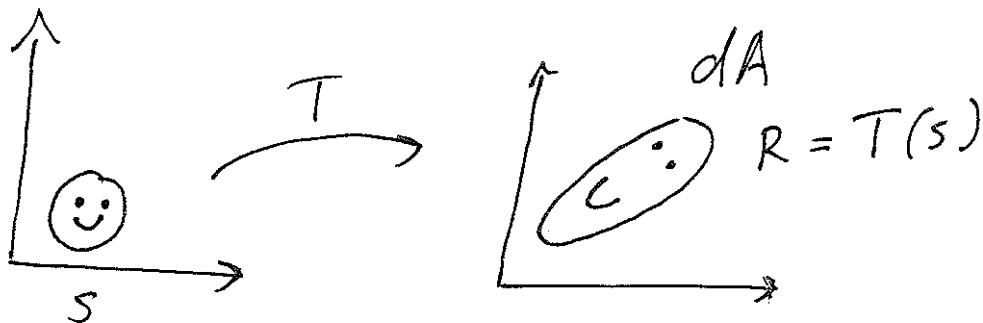
$$= \int_0^1 3u - 3u^2 du = \left. \frac{3}{2}u^2 - u^3 \right|_{u=0}^{u=1} = \frac{1}{2}$$

Fun check: Do as shown on pg 89.

In general, if $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear,

with $T(u, v) = (au + bv, cu + dv)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then



and $dA = \begin{vmatrix} a & b \\ c & d \end{vmatrix} du dv$.

Thus

$$\iint_R f(x, y) dA = \iint_S f(T(u, v)) \begin{vmatrix} a & b \\ c & d \end{vmatrix} du dv$$

Linear Approx: Recall if $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ is diff at

(u, v) then

$$g(u + \Delta u, v + \Delta v) = g(u, v) + g_u(u, v) \Delta u + g_v(u, v) \Delta v + \underbrace{E(\Delta u, \Delta v)}_{\text{small.}}$$

Now consider $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(u, v) = (g(u, v), h(u, v)).$$

Then

$$T(u+\Delta u, v+\Delta v) = T(u, v) + J_{(u,v)}(\Delta u, \Delta v) + \text{Error}$$

where

$J_{(u,v)}$ is the linear transformation with matrix $\begin{pmatrix} g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v) \end{pmatrix}$

Reason

$$T(u+\Delta u, v+\Delta v)$$

$$\approx \begin{pmatrix} g(u,v) + g_u(u,v)\Delta u + g_v(u,v)\Delta v, \\ h(u,v) + h_u(u,v)\Delta u + h_v(u,v)\Delta v \end{pmatrix}$$

Thus locally T looks like a linear transformation...

— to be continued —