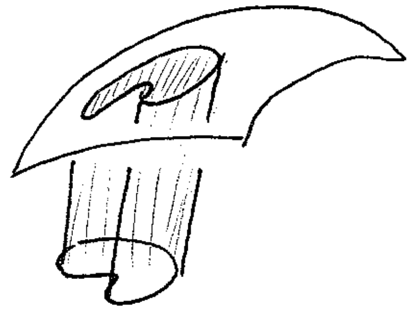
Lecture 24: Triple integrals (§15.6)

Previously:  Region in $\mathbb{R}^2$
$f: \mathbb{R}^2 \to \mathbb{R}$



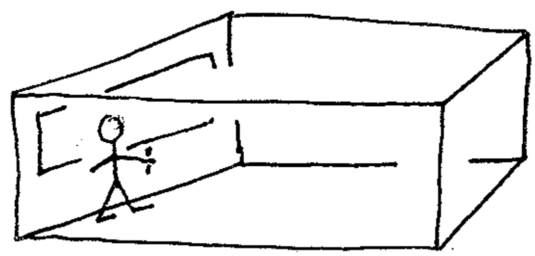
$$\iint_R f\, dA \longrightarrow \text{Volume of}$$

$$\longrightarrow \iint_R 1\, dA = \text{Area of } R$$

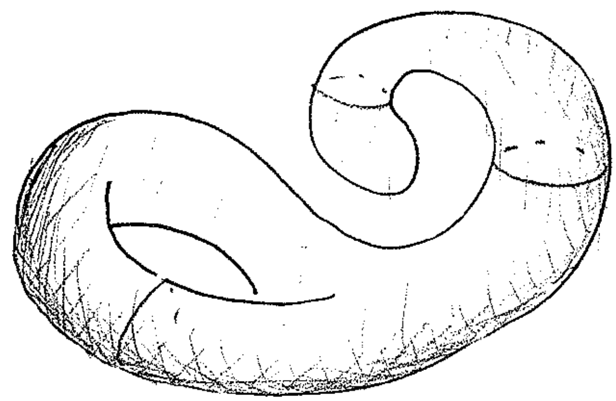
$$\longrightarrow \frac{1}{\text{Area}} \iint_R f\, dA = \text{Average of } f \text{ on } R.$$

[Also total mass, center of mass, etc...]

Triple Ints: R a region in $\mathbb{R}^3$



$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$



$$\iiint_R f(x,y,z)\, dV$$

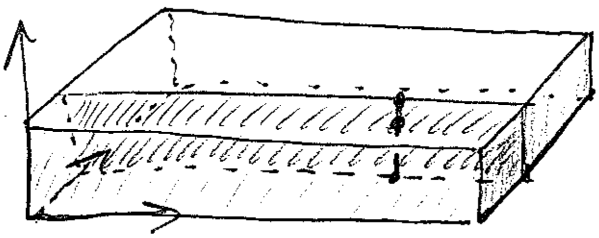
$$\iiint_R 1\, dV = \text{Volume}$$

$$\text{Average} = \frac{1}{\text{Vol}} \iiint_R f\, dV$$

$$\text{Mass} = \iiint_R \underbrace{\rho(x,y,z)}_{\text{density}}\, dV$$

As always, defined math. in terms of certain Riemann sums...

Ex:



First slice by y, then x then z.

$$0 \le x \le 3$$
$$0 \le y \le 2$$
$$0 \le z \le 1$$

$$\iiint_R f\,dV = \int_0^2\int_0^3\int_0^1 xy+z \; dz\,dx\,dy$$

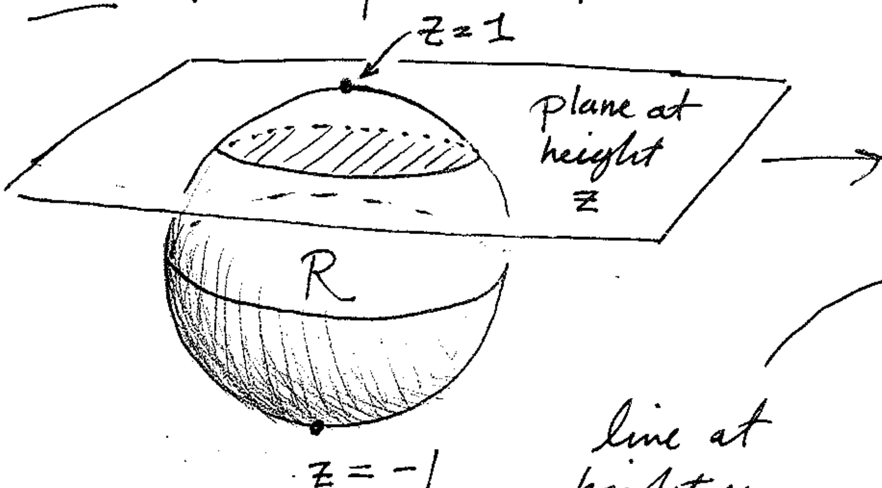
$$= \ldots = 12.$$

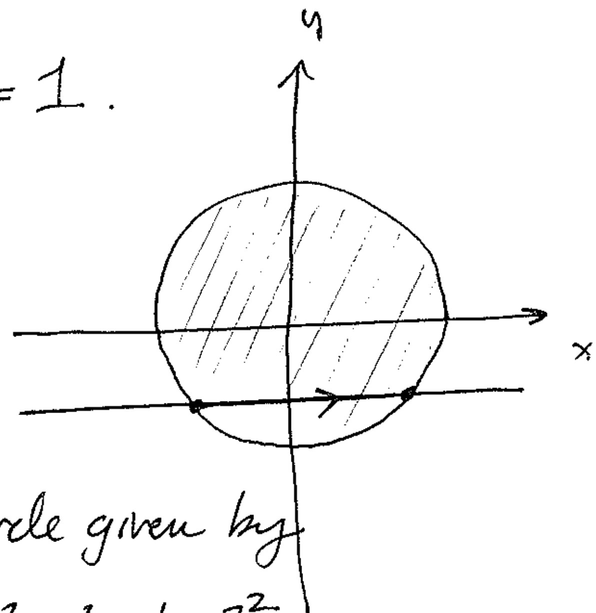
$$f(x,y,z) = xy+z$$

Just like double ints

[Of course, you'd get the same answer any of the other five ways.]

Ex: Volume of unit sphere $x^2+y^2+z^2=1$.





line at height y.

circle given by $x^2+y^2=1-z^2$

y goes from $-\sqrt{1-z^2}$ to $\sqrt{1-z^2}$

x goes from $-\sqrt{1-y^2-z^2}$ to $\sqrt{1-y^2-z^2}$

$$\iiint_R 1 \; dV = \underbrace{\int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} 1 \, dx \, dy}_{} \, dz$$

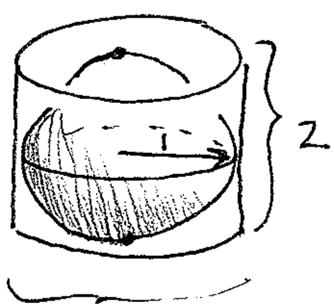
Can do with trig substitution / table

Or note that it is the area of a disc of radius $\sqrt{1-z^2}$

$$= \int_{-1}^{1} \pi (1-z^2)\, dz = \pi\left(z - \frac{z^3}{3}\right)\Bigg|_{z=-1}^{1}$$

$$= \pi\left(\left(1 - 1/3\right) - \left(-1 + 1/3\right)\right) = \frac{4}{3}\pi.$$

Cf Archimeades.

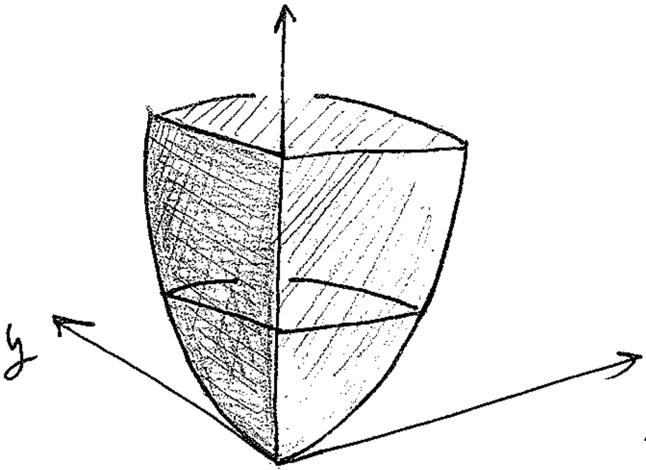


Base area $= \pi$

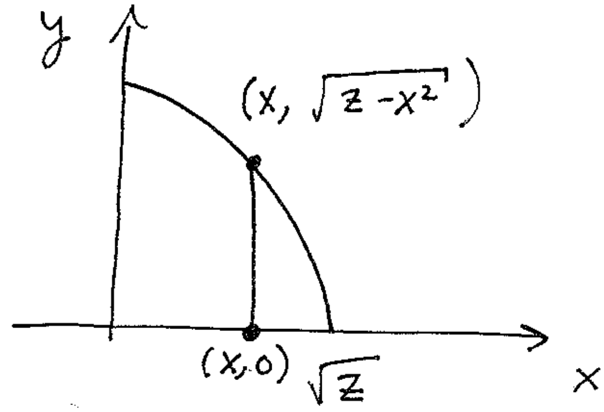
$$\Rightarrow \mathrm{Vol}(\square) = 2\pi$$

So $\dfrac{\mathrm{Vol}(\square)}{\mathrm{Vol}(\ominus)} = \dfrac{3}{2}$

Ex: Consider the region in the pos octant bounded by the planes $x=0$, $y=0$, $z=2$ and the surface $z = x^2+y^2$.



Slice by planes of fixed z-height to get



$$\iiint_R f\, dV = \int_0^2 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-x^2}} f(x,y,z)\, dy\, dx\, dz$$

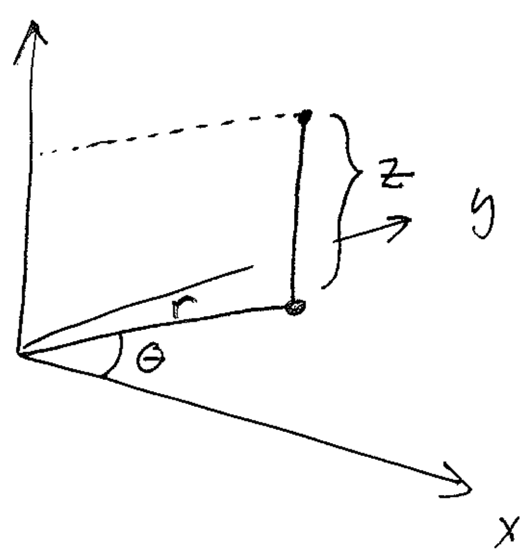
Alternate coordinates: (15.7 and 15.8)

Cylindrical:

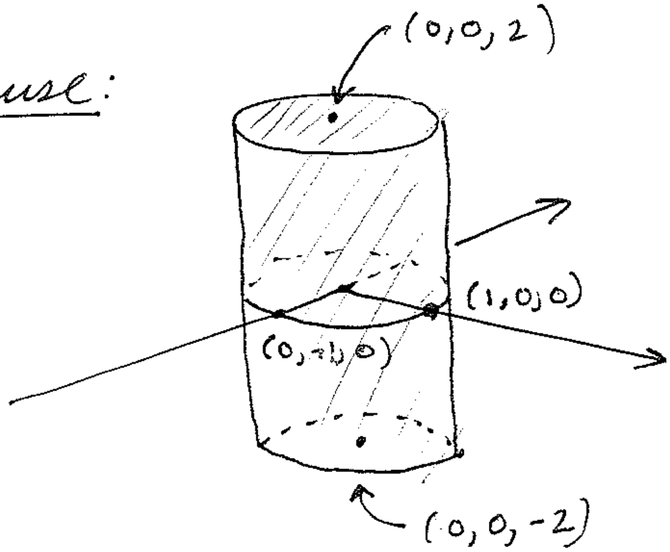
$$\begin{array}{ccc} & & x \quad\quad y \quad\quad z \\ (r,\theta,z) & \longleftrightarrow & (r\cos\theta, r\sin\theta, z) \end{array}$$

$0 \leq r$

$0 \leq \theta \leq 2\pi$



Typical use:



Rectangular

$$-2 \le z \le 2$$

$$x^2 + y^2 \le 1$$

Cylindrical:

$$0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

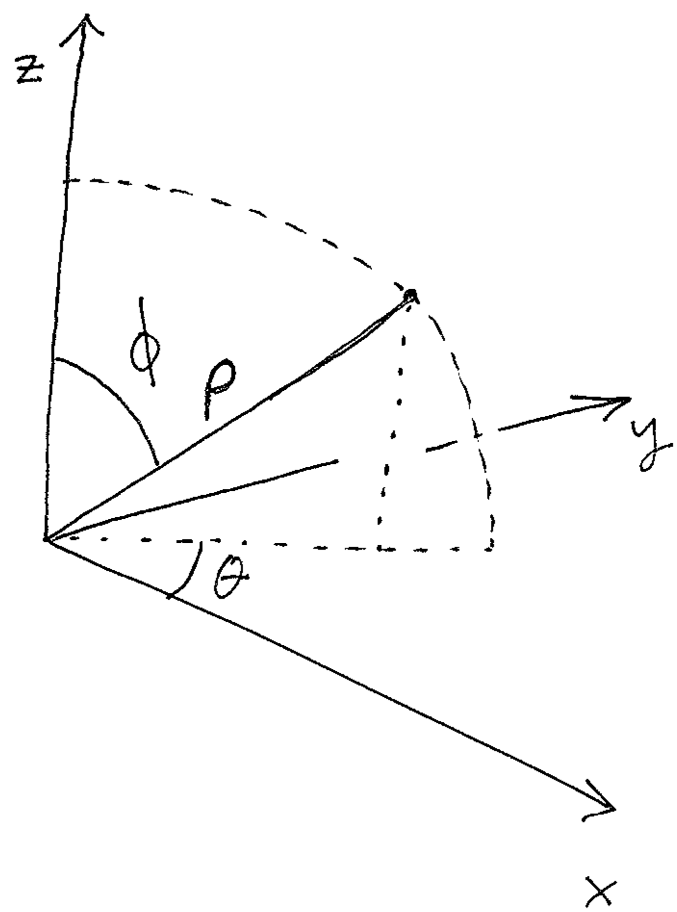
$$-2 \le z \le 2.$$

Spherical:

$$0 \le \rho$$

$$0 \le \theta \le 2\pi$$

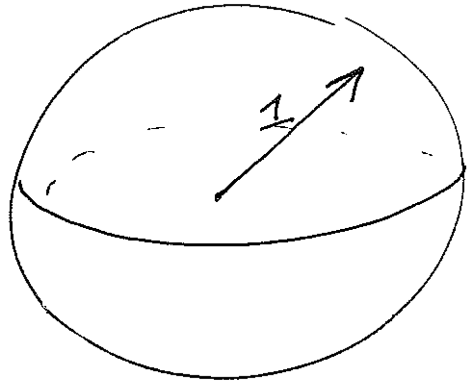
$$0 \le \phi \le \pi$$



$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$ ← start here.



<u>Rectangular:</u>

$-1 \le z \le 1$

$-\sqrt{1-z^2} \le y \le \sqrt{1-z^2}$

$-\sqrt{1-y^2-z^2} \le x \le \sqrt{1-y^2-z^2}$

<u>Cylindrical:</u>

$-1 \le z \le 1$

$0 \le r \le \sqrt{1-z^2}$

$0 \le \theta \le 2\pi$

<u>Spherical:</u>

$0 \le \rho \le 1$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi.$

<u>Q</u>: How do we integrate in these different coordinates?