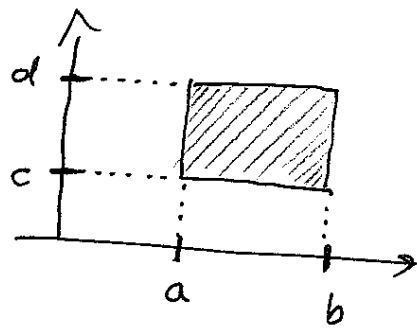
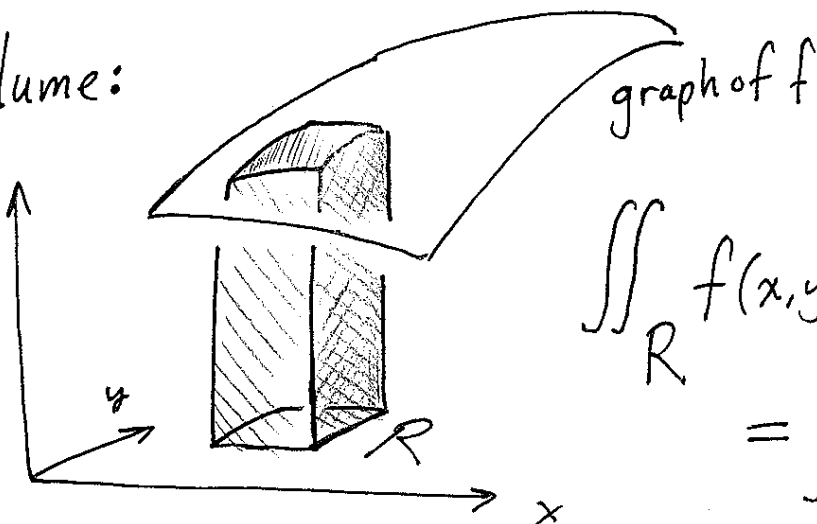


Lecture 22: Integrating over more complicated regions (§15.2-3)

Last time: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ $R = \text{rectangle in } \mathbb{R}^2$

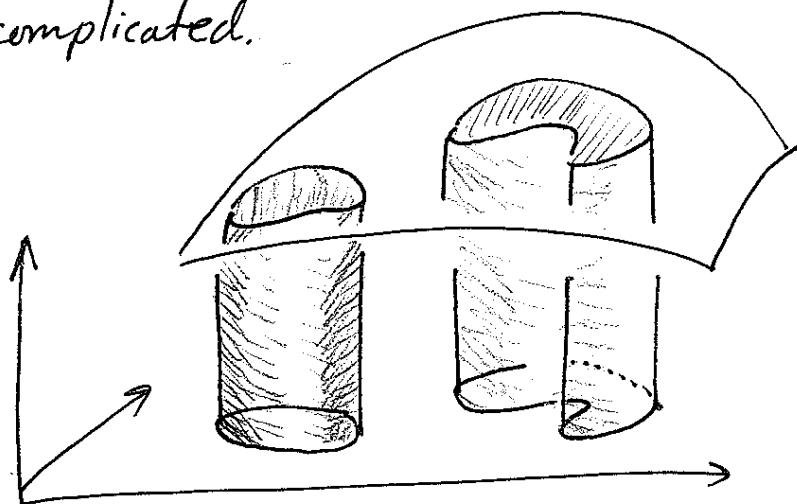
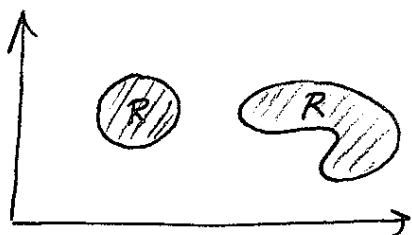
Volume:



$$\iint_R f(x,y) dA = \int_c^d \left(\int_a^b f(x,y) dx \right) dy$$

[Other interpretations: averages, total mass, center of mass, ...]

Today: When R is more complicated.

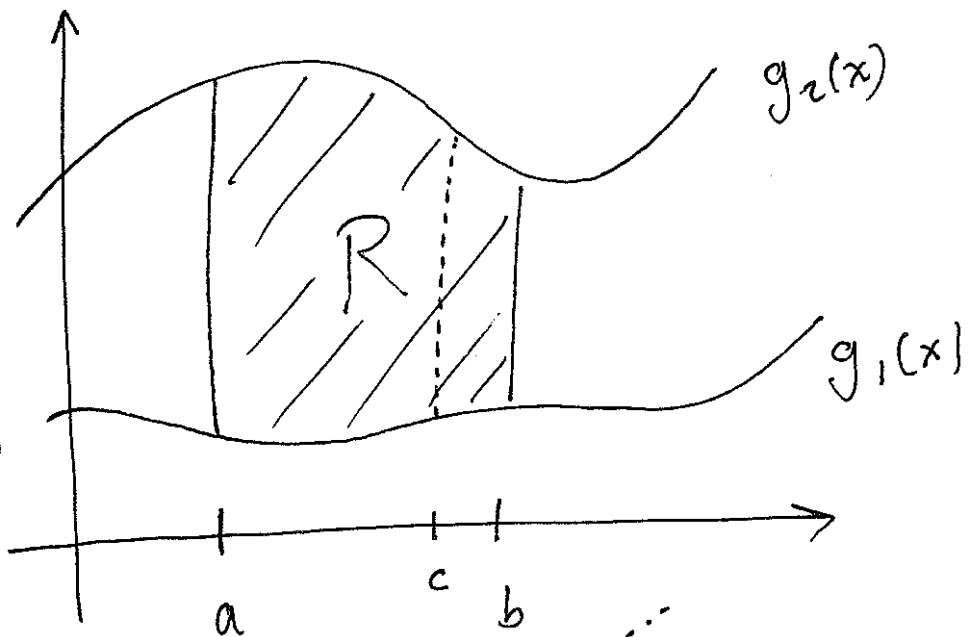


Q: How to compute

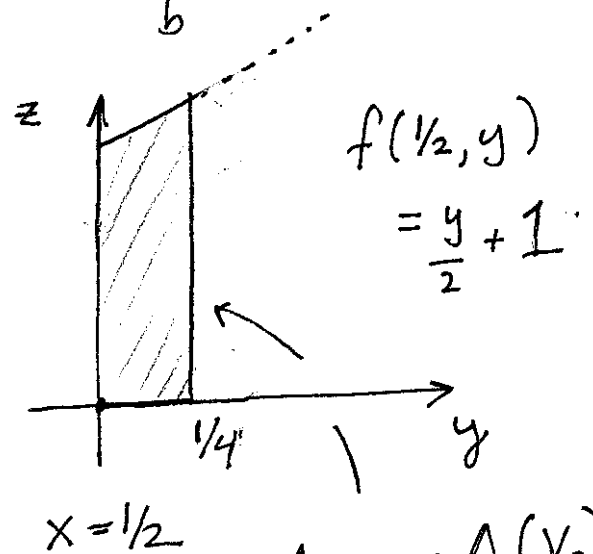
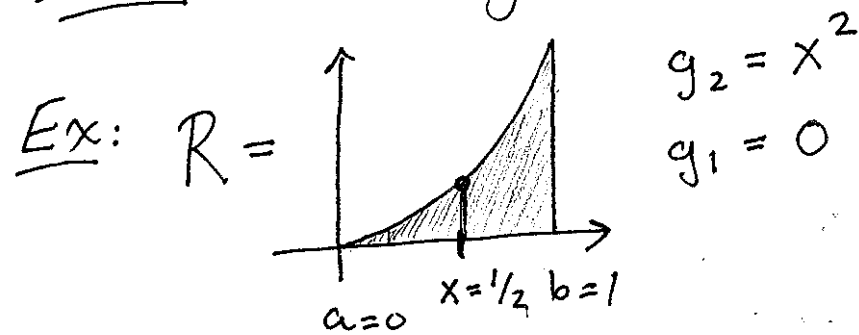
$$\text{Volume} = \iint_R f(x,y) dA ?$$

Suppose R
has the form:

$$R = \left\{ a \leq x \leq b \text{ and } \right. \\ \left. g_1(x) \leq y \leq g_2(x) \right\}$$



Idea: Slice along lines $x=c$.



Area = $A(1/2)$

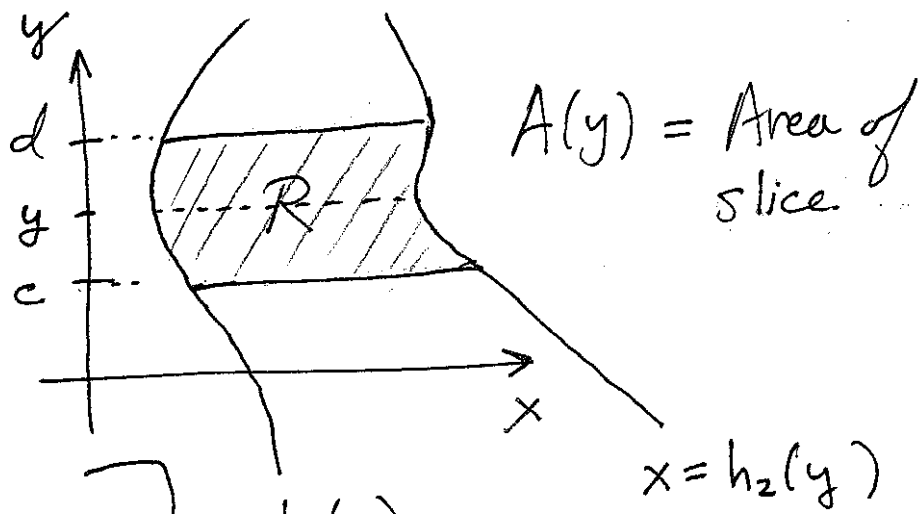
$$\iint_R \underbrace{xy + 1}_{f(x,y)} dA = \int_0^1 A(x) dx$$

$$= \int_0^1 \left(\int_0^{x^2} xy + 1 dy \right) dx = \int_0^1 \left(\frac{xy^2}{2} + y \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_0^1 \frac{1}{2} x (x^2)^2 + x^2 dx = \int_0^1 \frac{1}{2} x^5 + x^2 dx$$

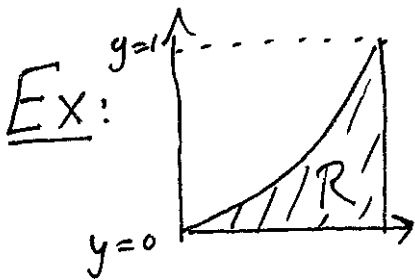
$$= \frac{1}{12} x^6 + \frac{x^3}{3} \Big|_{x=0}^1 = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

Similar case:



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$$\begin{aligned} \iint_R f(x,y) dA &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy \\ &= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y) dx \right) dy \end{aligned}$$

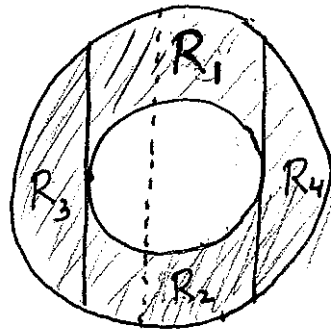


$$\begin{aligned} h_1(y) &= \sqrt{y} \\ h_2(y) &= 1 \end{aligned}$$

$$\begin{aligned} \iint_R xy + 1 dA &= \int_0^1 A(y) dy = \int_0^1 \left(\int_{\sqrt{y}}^1 xy + 1 dx \right) dy \\ &= \int_0^1 \left(\frac{1}{2} x^2 y + x \Big|_{x=\sqrt{y}}^{x=1} \right) dy = \int_0^1 \left(\frac{1}{2} y + 1 \right) - \left(\frac{1}{2} y^2 + \sqrt{y} \right) dy \\ &= \int_0^1 \left(-\frac{1}{2} y^2 + \frac{1}{2} y - \sqrt{y} + 1 \right) dy = \left. -\frac{y^3}{6} + \frac{y^2}{4} - \frac{2}{3} y^{3/2} + y \right|_{y=0}^1 \\ &= -\frac{1}{6} + \frac{1}{4} - \frac{2}{3} + 1 = \frac{-2 + 3 - 8 + 12}{12} = \frac{5}{12} \checkmark \end{aligned}$$

Book calls these two kinds of regions type I and type II. Some regions are both, in which case it can be easier to do things one way or the other.

General region:

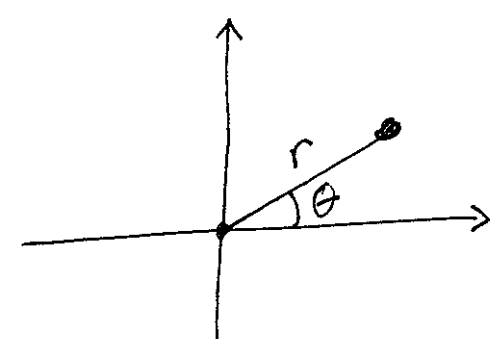


$$\iint_R f dA = \sum_{i=1}^4 \iint_{R_i} f dA$$

(A) Cut into simple pieces.

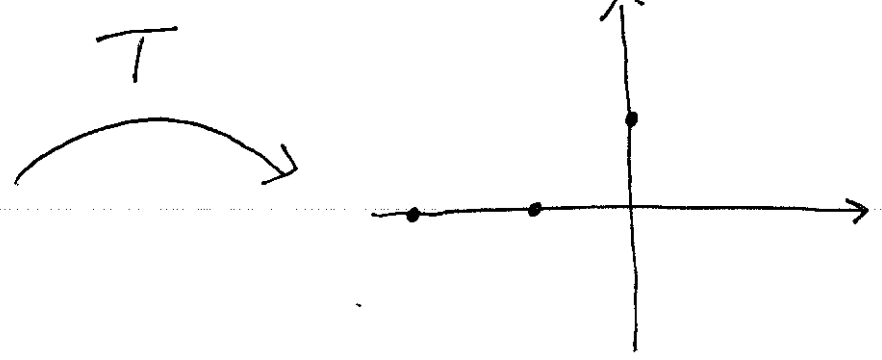
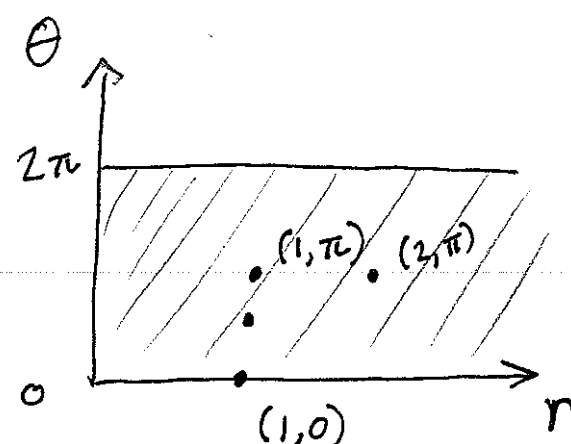
(B) Change Coordinates, so R becomes easier to describe.

Polar Coordinates:



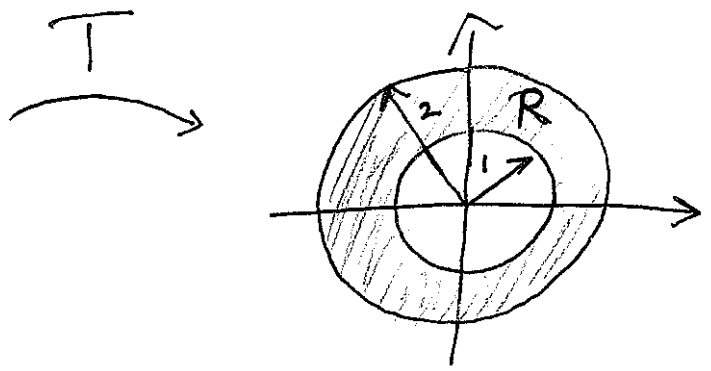
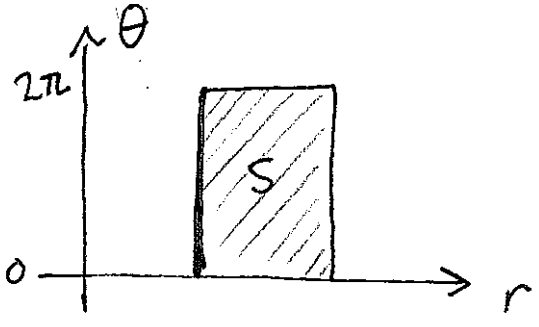
$$x = r \cos \theta$$

$$y = r \sin \theta$$



$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$T(r, \theta) = (r \cos \theta, r \sin \theta)$$



$$S = \left\{ \begin{array}{l} 1 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$$

$$R = T(S)$$

Goal: Relate $\iint_R f \, dA$ to an integral over S .

First try:

$$\begin{aligned} \iint_R 1 \, dA &\stackrel{\text{Guess!}}{=} \iint_S 1 \, dA = \int_0^{2\pi} \int_1^2 1 \, dr \, d\theta \\ &= \int_0^{2\pi} r \Big|_{r=1}^2 \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi. \end{aligned}$$

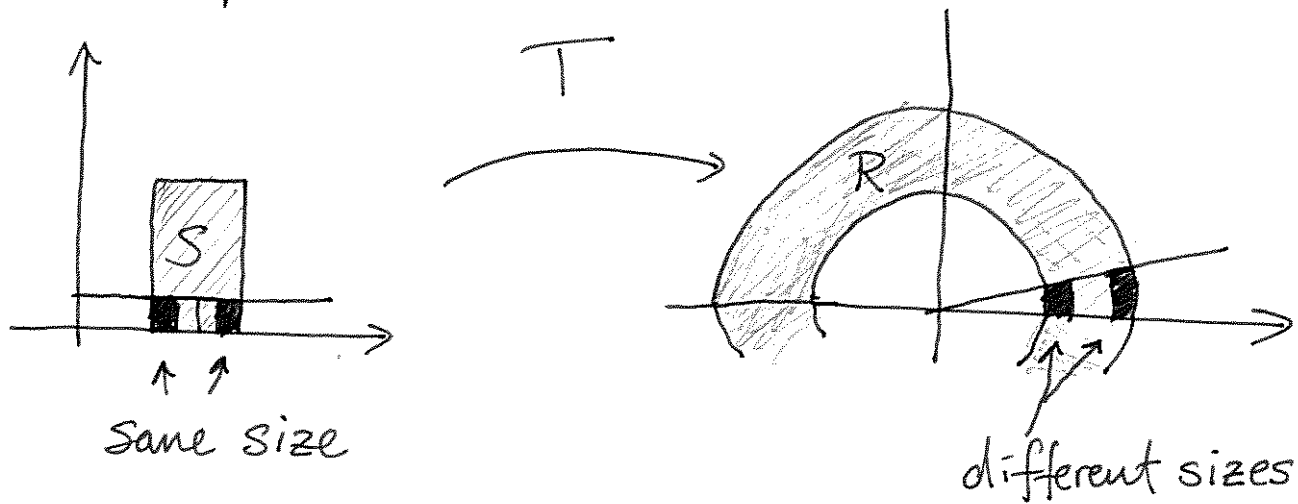
Is this right?

$$\begin{aligned} \iint_R 1 \, dA &= \text{Area}(R) = \text{Area}(\text{disc of rad } 2) - \text{Area}(\text{disc of rad } 1) \\ &= \pi 2^2 - \pi 1^2 = 3\pi. \end{aligned}$$

So this didn't work...

Source of problem: T distorts area...
(in a non uniform way).

For example:



Solution: In polar coordinates $dA = r dr d\theta$

[Will explain in detail next lecture, but for now]
[let's just see that it works in this case.]

$$\begin{aligned}\iint_S 1 dA &= \int_0^{2\pi} \int_1^2 1 \cdot r dr d\theta = \int_0^{2\pi} \left(\frac{1}{2} r^2 \Big|_{r=1}^{r=2} \right) d\theta \\ &= \int_0^{2\pi} \frac{1}{2} (2^2 - 1^2) d\theta = \frac{3}{2} \int_0^{2\pi} 1 d\theta \\ &= \frac{3}{2} \cdot 2\pi = 3\pi\end{aligned}$$

which is indeed the same as $\iint_R 1 dA$!