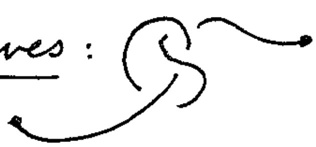
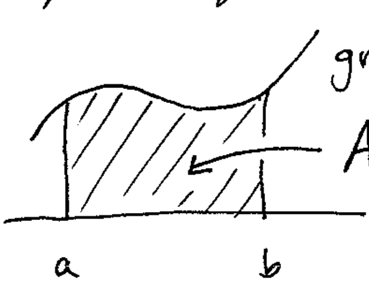
Lecture 21: Multivariable Integration (§15.1)

<u>Curves</u>: 

$\int_C f\,ds \quad \int_C \vec{F}\cdot d\vec{r}$

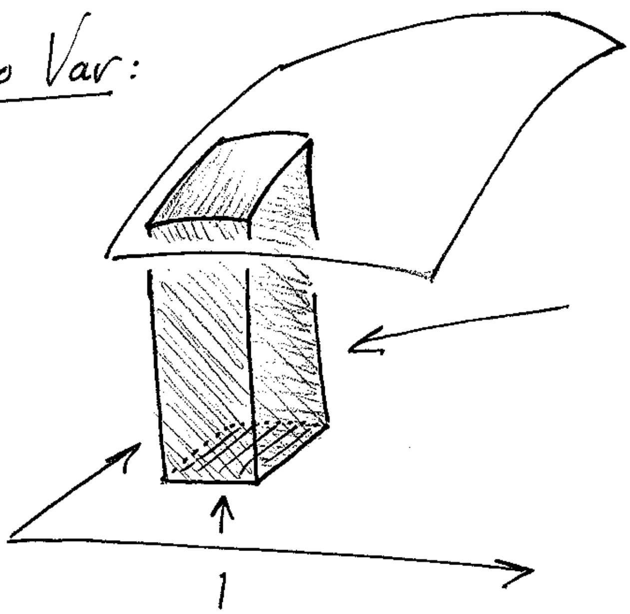
<u>Goal</u>: Surfaces  $\int_S f\,dA$

[Things like parametrization, integration will repeat for surfaces...]

<u>One Var</u>: 

graph of f

Area $= \int_a^b f(x)\,dx$ [Computed using the Fund. Thm. of Calc.]

<u>Two Var</u>: 

graph of $f: \mathbb{R}^2 \to \mathbb{R}$

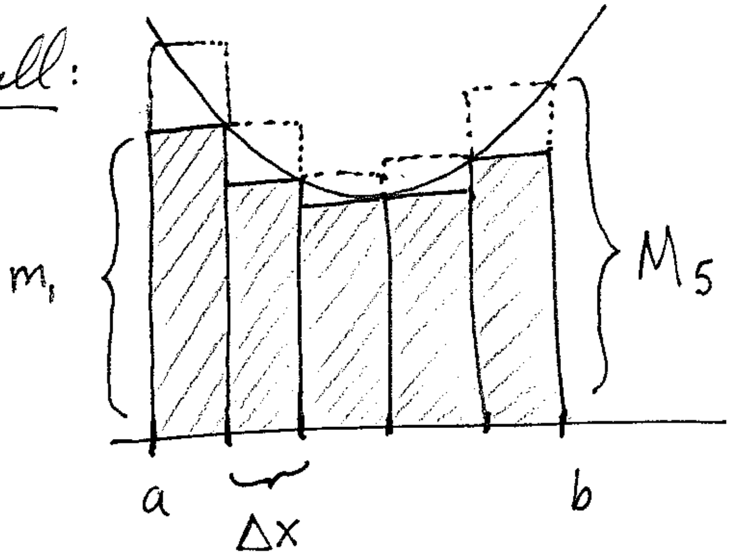
Volume is $\iint_R f(x,y)\,dA$

Base R, a square

<u>Q1</u>: How do we formulate volume mathematically?

<u>Q2</u>: How do we compute it?

Recall:



m_i = min of f on i^{th} subinterval

M_i = max of f on i^{th} subinterval.

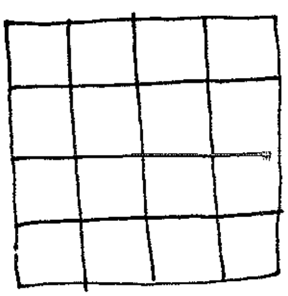
Then

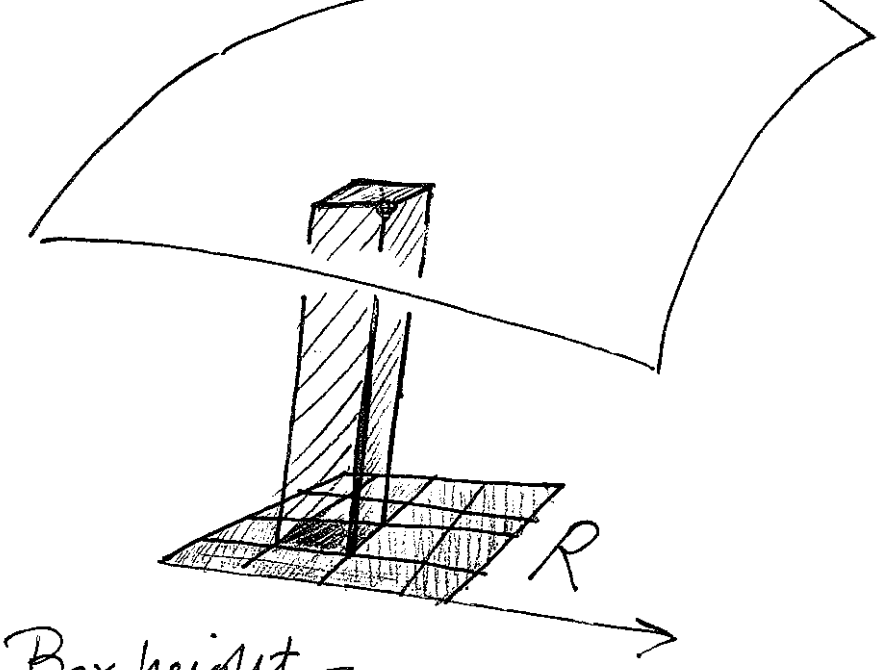
$$\sum_{i=1}^{n} m_i \Delta x \le \int_a^b f(x)\,dx \le \sum_{i=1}^{n} M_i \Delta x$$

If f is continuous, then as $\Delta x \to 0$ the two bounds converge to the thing in the middle.

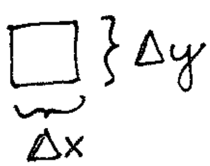
Two Var

R:





Each square



has area $\Delta x \Delta y$.

Box height = min of f on the subsquare.

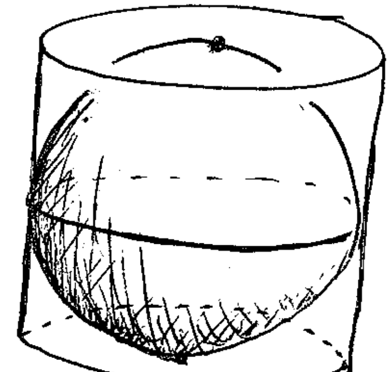
Thus

$$\sum_{\substack{\text{small}\\\text{squares}}} \begin{pmatrix}\text{min of } f\\ \text{on subsquare}\end{pmatrix} \Delta x \Delta y \leq \iint_R f(x,y)\, dA$$

$$\leq \sum_{\substack{\text{small}\\\text{squares}}} \begin{pmatrix}\text{max of } f\\ \text{on subsquare}\end{pmatrix} \Delta x \Delta y$$

As $\Delta x, \Delta y \to 0$ these bounds converge to define the integral. [Provided f is continuous.]

Q2: How do we compute it?



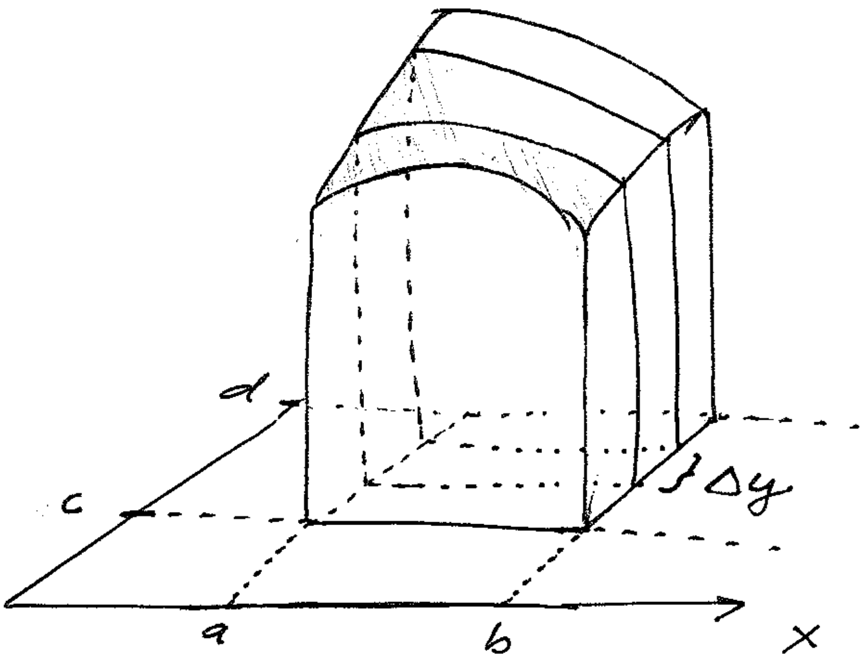
<u>Archimedes</u> (225 B.C.E.)

3 : 2

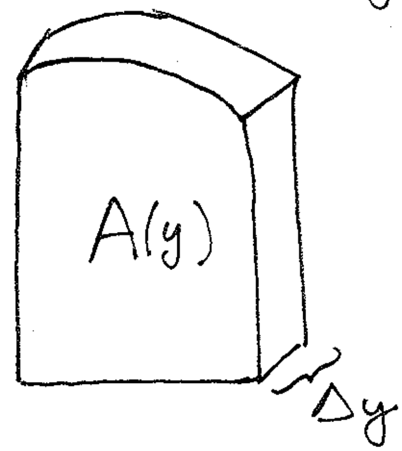
volume <u>and</u> surface area!

<u>Key</u>: Reduce to one var integrals by slicing.

Let $A(y)$ be the area of the cross section with the given y coor.



Volume of a slice is $\approx A(y)\,\Delta y$



Add to get total volume

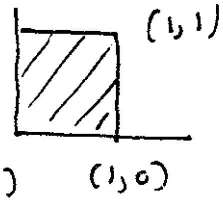
So
$$\iint_R f(x,y)\,dA = \int_c^d A(y)\,dy$$

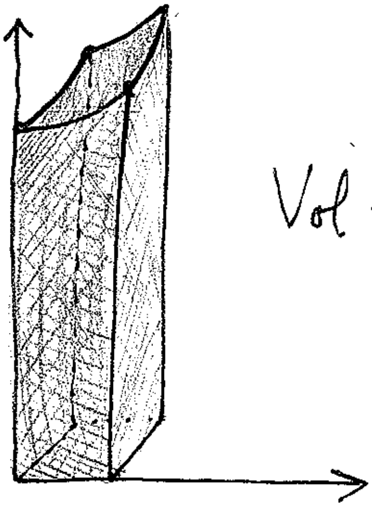
y is fixed

$$= \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Ex: $f(x,y) = x^2+y^2+5 \quad R =$



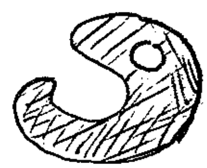


$$\text{Vol} = \iint f(x,y)\, dA = \int_0^1 \left(\int_0^1 (x^2+y^2+5)\, dx \right) dy$$

$$= \int_0^1 \left(\frac{x^3}{3} + (y^2+5)\,x \Big|_{x=0}^{x=1} \right) dy$$

$$= \int_0^1 \frac{1}{3} + (y^2+5)\, dy = \frac{16}{3} y + \frac{y^3}{3} \Big|_{y=0}^{y=1} = \frac{17}{3} = 5\frac{2}{3}$$

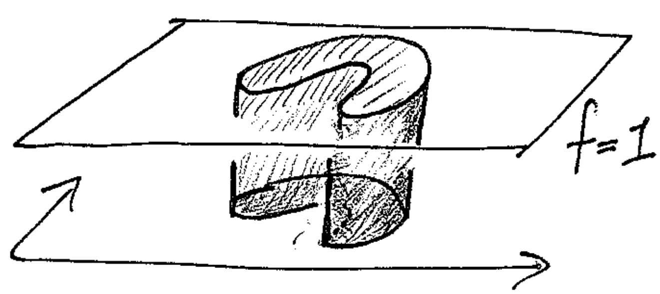
Next time: Dealing with regions R which are not rectangles



Idea: Break into small boxes.

Other meanings of the integral:

① $\iint_R 1\, dA = \text{Area of } R$



$\text{Vol} = (\text{height}) \cdot (\text{Area})$

$$\boxed{dA = d(\text{Area})}$$

② Average of $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ on R is

$$\text{Average} = \frac{1}{\text{Area}(R)} \iint_R f \, dA$$

③ R made of material with density given by $\rho: \mathbb{R}^2 \to \mathbb{R}$. Then

$$\text{Total Mass} = \iint_R \rho \, dA.$$