Lecture 20: Conservative vector fields II (§16.3)

Last time: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$.

Thm A: A vector field $\vec{F}$ on an open connected region D in $\mathbb{R}^2$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

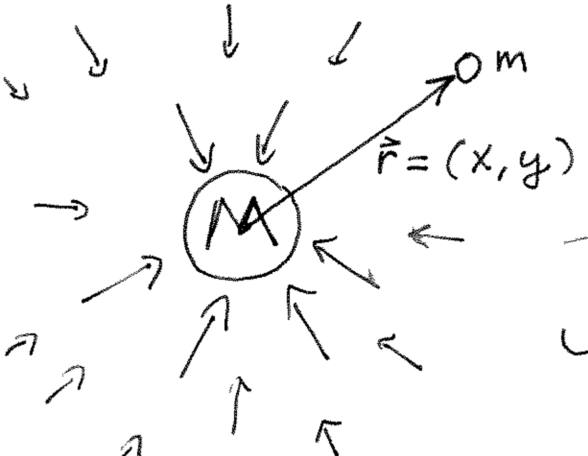
Thm B: $\vec{F} = (P, Q)$ on a simply connected open D in $\mathbb{R}^2$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

Thm A Also works for $\mathbb{R}^n$.

Thm B Has analogs for $\mathbb{R}^n$, but more complicated.

[We will come back to this later, but see §16.3 #29
Note to self: condition is $H^1(D; \mathbb{R}) = 0$.]

Physical motivation: Force $\vec{F} = -\frac{MmG}{|\vec{r}|^3} \vec{r}$

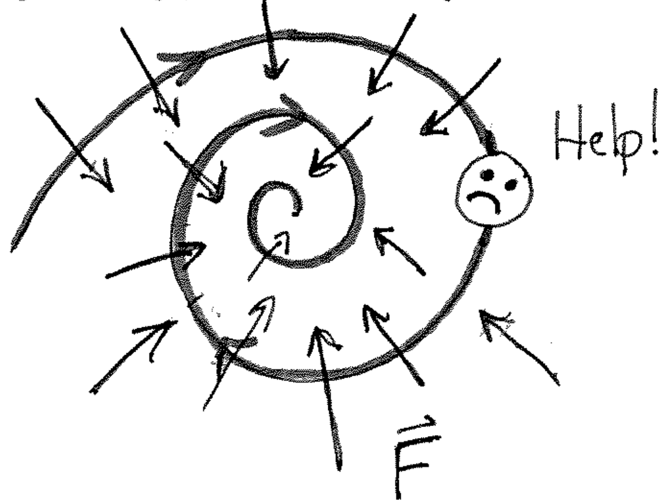


$V = -\frac{MmG}{|\vec{r}|}$ Potential function

Note: $\vec{F} = -\nabla V$

V is larger the farther away m is and $\vec{F}$ points in the direction of fastest <u>decrease</u> of V.

More generally, suppose an object's path is $\vec{r}: \mathbb{R} \to \mathbb{R}^2$, acted on by a conservative force $\vec{F} = -V$ for $V: \mathbb{R}^2 \to \mathbb{R}$.



<u>Newton's Law</u>: $F = ma$

or

$$\vec{F}(\vec{r}(t)) = m\vec{r}''(t)$$

<u>Total Energy</u>:

$$E(t) = \begin{pmatrix}\text{Kinectic} \\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{Potential} \\ \text{energy}\end{pmatrix}$$

$$= \frac{1}{2} m \, |\vec{r}'(t)|^2 + V(\vec{r}(t))$$

<u>Conservation of Energy</u>: $E(t)$ is constant, independent of t.

<u>Reason</u>: Let's compute $E'(t)$. If $\vec{r} = (r_1, r_2)$,

Then

$$\frac{d}{dt}\left(\frac{1}{2}m|\vec{r}'(t)|^2\right) = \frac{1}{2}m\frac{d}{dt}\left((r_1'(t))^2 + (r_2'(t))^2\right)$$

$$= \frac{1}{2}m\left(2r_1'(t)r_1''(t) + 2r_2'(t)r_2''(t)\right)$$

$$= m\,\vec{r}''(t)\cdot\vec{r}'(t)$$

Also

$$\frac{d}{dt}V(\vec{r}(t)) = \frac{\partial V}{\partial x}(\vec{r}(t))\,r_1'(t) + \frac{\partial V}{\partial y}(\vec{r}(t))\,r_2'(t)$$

$$= \nabla V(\vec{r}(t))\cdot\vec{r}'(t)$$

So

$$E'(t) = m\vec{r}''(t)\cdot\vec{r}'(t) + \nabla V(\vec{r}(t))\cdot\vec{r}'(t)$$

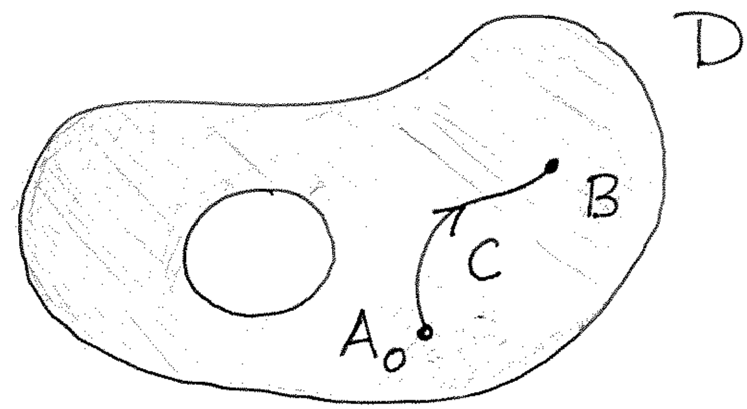
$$= \vec{F}(\vec{r}(t))\cdot\vec{r}'(t) - \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)$$

$$= 0$$

Hence energy is conserved!

Thm A: A vector field $\vec{F}$ on an open connected region D in $\mathbb{R}^2$ is conservative if and only if $\int_C \vec{F}\cdot d\vec{r}$ is path independent.

<u>Reason for Thm A</u>: Suppose $\int_C \vec{F}\cdot d\vec{r}$ is path independent. Pick a point A_0 in D.



Define $f: D \to \mathbb{R}$ by

$$f(B) = \int_C \vec{F}\cdot d\vec{r}$$

where C is any path from A_0 to B inside of D.

[This is an odd sort of function, but it is a way of associating a number to each point...]

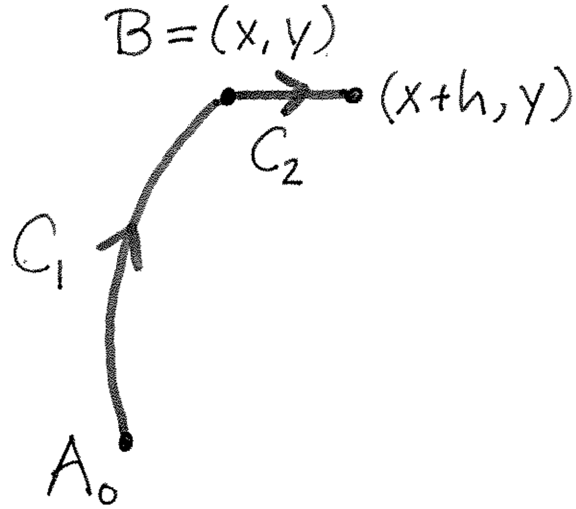
<u>Note</u>: $f(A_0) = 0$ and so, the F.T.L.I. says

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A_0) = f(B)$$

<u>Point</u>: $\nabla f = \vec{F}$. For instance, let's compute

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

Now consider how we compute these:



$$f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$f(x+h,y) = \int_{C_1 + C_2} \vec{F} \cdot d\vec{r}$$

Thus $$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{1}{h} \int_{C_2} \vec{F} \cdot d\vec{r}$$

Now if $\vec{F} = (P, Q)$ then as $\vec{T} = (1, 0)$ we have

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot \vec{T} \, ds = \int_{C_2} P \, ds$$

So

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \underbrace{\frac{1}{h} \int_{C_2} P \, ds}_{\text{Average of } P \text{ on } C_2} = P(x,y)$$

As averaging over shorter and shorter curves.

Likewise, $\frac{\partial f}{\partial y} = Q$ and so $\nabla f = \vec{F}$, that is $\vec{F}$ is conservative.