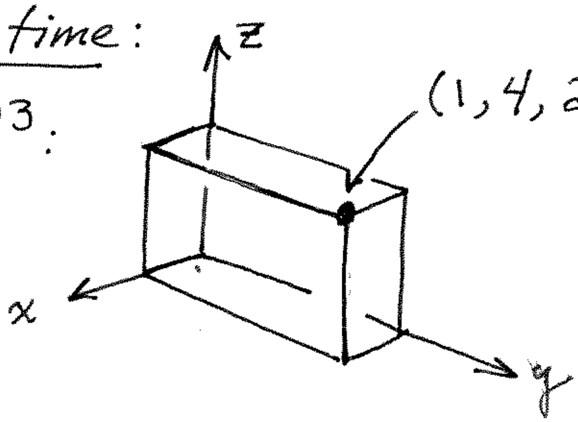
Lecture 2: Vectors (§12.2) and the dot product (§12.3)

Last time:

$\mathbb{R}^3$:



$\mathbb{R}^n$: tuples $(x_1, x_2, \ldots, x_n)$ of real numbers

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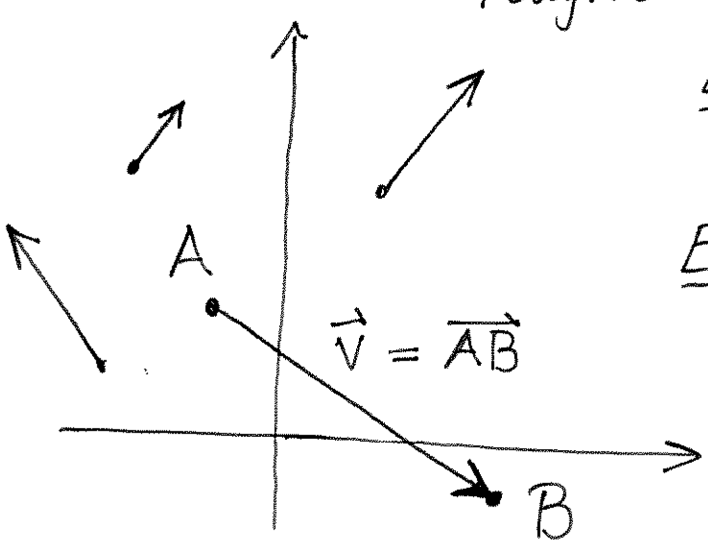
Vectors in $\mathbb{R}^2$: Arrows where both direction and length are important.

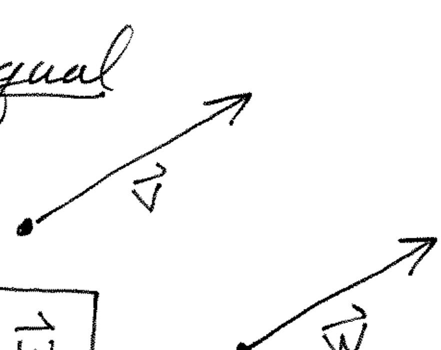


Ex: Plot windspeed and direction.

Ex: Record relative position of two points

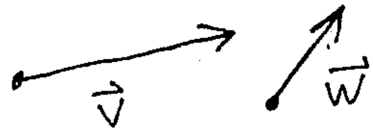
Denoted $\vec{v}$ or $\vec{w}$ or similar. [Bold used in books.]

Vectors $\vec{v}$ and $\vec{w}$ are regarded as equal if they are the same up to translation.

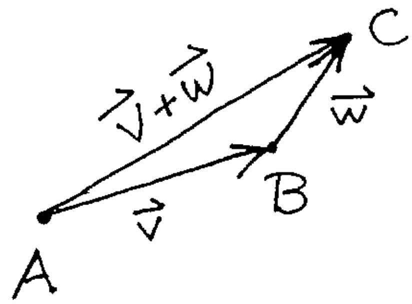


$$\boxed{\vec{v} = \vec{w}}$$

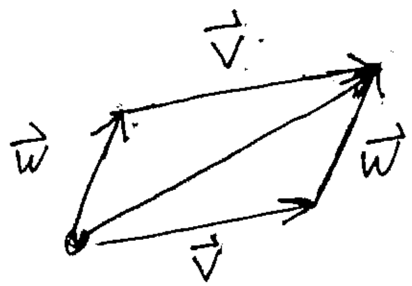
Addition:



Start at A, head along $\vec{v}$ to B, then along $\vec{w}$ to C.



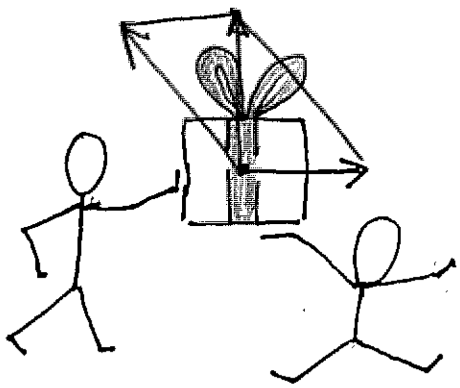
Note:



and so

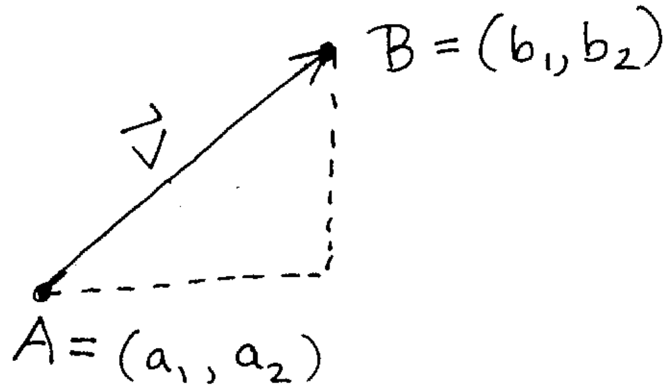
$$\boxed{\vec{v}+\vec{w} = \vec{w}+\vec{v}}$$

Ex: Force



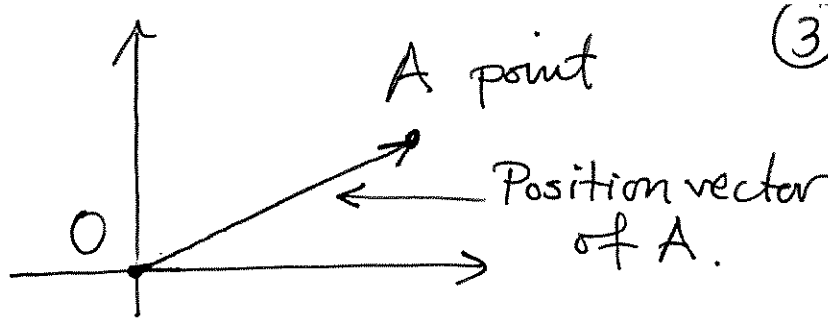
As with points in $\mathbb{R}^2$, vectors are determined by a pair of numbers

$$\vec{v} = (b_1 - a_1,\ b_2 - a_2)$$



Still, points and vectors are different: absolute vs. relative positions.

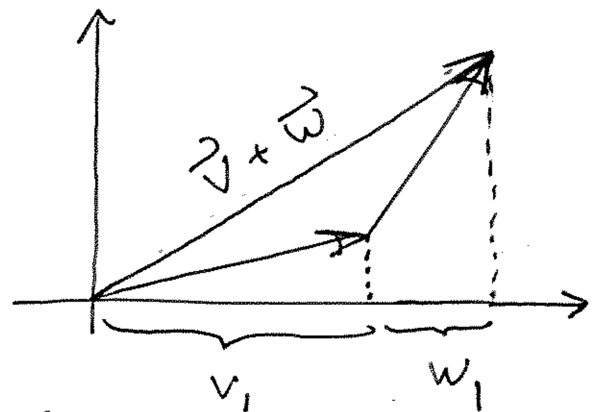
A common way to relate them



Formula for addition:

If $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ then

$$\vec{v} + \vec{w} = (v_1 + w_1 , v_2 + w_2).$$

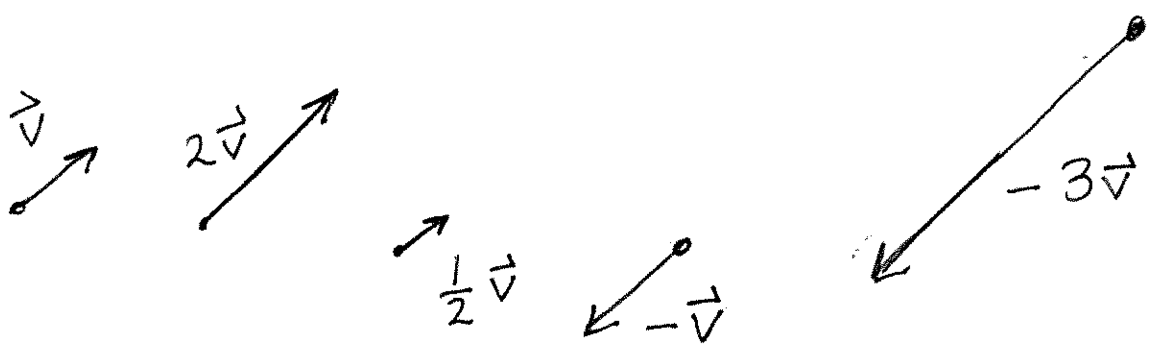


Scalar Multiplication:

vector

$c\,\vec{v}$ = vector with same direction but with length scaled by c.

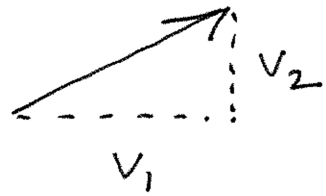
number



In coordinates, if $v = (v_1, v_2)$ then

$$c\,\vec{v} = (c v_1 , c v_2)$$

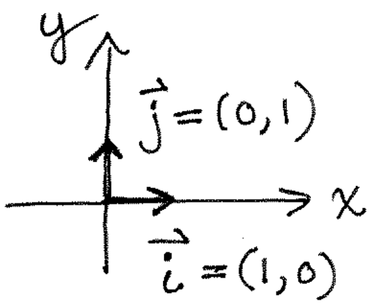
Length: $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

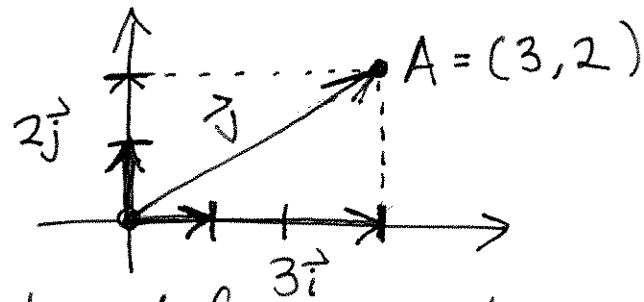


Some books use $\|\vec{v}\|$.

Standard Vectors: $\vec{v} = 3\vec{i} + 2\vec{j} = 3\cdot(1,0) + 2\cdot(0,1)$

$= (3,0) + (0,2) = (3,2)$





Properties: [Can work out from geometry or algebra.]

$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ $\quad (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

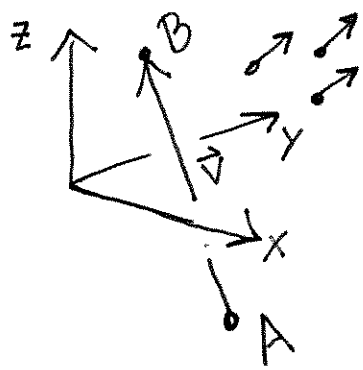
$\vec{v} + \vec{0} = \vec{v}$ if $\vec{0} = (0,0)$ $\quad \vec{v} + (-1)\vec{v} = \vec{0}$

$c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ $\quad (c+d)\vec{v} = c\vec{v} + d\vec{v}$

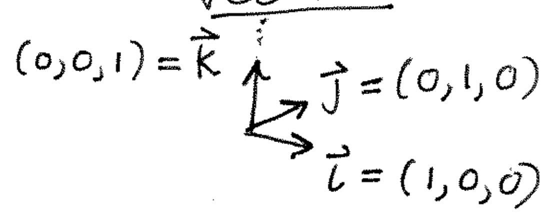
$(cd)\vec{v} = c(d\vec{v})$ $\quad 1\vec{v} = \vec{v}$

Vectors in $\mathbb{R}^3$:

$\vec{v} = (v_1, v_2, v_3)$



Standard Vectors:



Vectors in $\mathbb{R}^n$: Once more, with feeling.

(See also, Math 415/416)

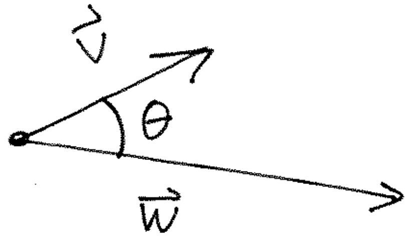
[Can we multiply vectors? In $\mathbb{R}^3$ there are two ways to do this, different from each other an mult. of numbers.]

Dot Product: $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ vectors in $\mathbb{R}^3$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

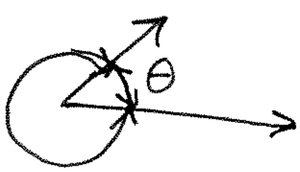
Ex: $\vec{v} = (1, 2, 0)$, $\vec{w} = (-1, 0, 2)$ $\quad \vec{v} \cdot \vec{w} = 1 \cdot (-1) + 2 \cdot 0 + 0 \cdot 2 = -1$

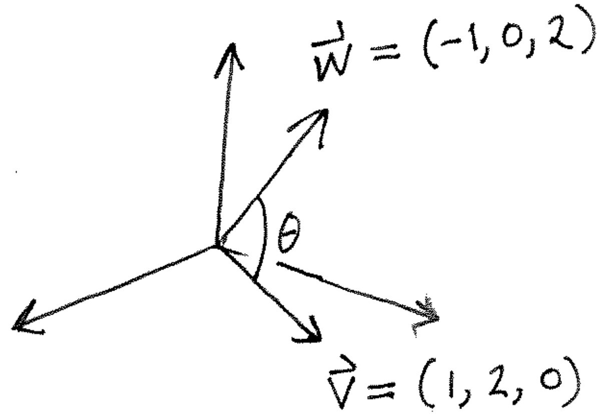
Key:



$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta}$$

Here $0 \leq \theta \leq \pi$ is the smaller of the two angles.





Ex: $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}$

$$= \frac{-1}{\sqrt{5} \cdot \sqrt{5}} = -\frac{1}{5} \Rightarrow \theta \approx 101.54°$$

Note: $\vec{v}\cdot\vec{w} = 0$ exactly when $\cos\theta = 0$, i.e. $\theta = \pi/2$ and the vectors meet at a right angle.

Properties: [Easy to see from the definition]

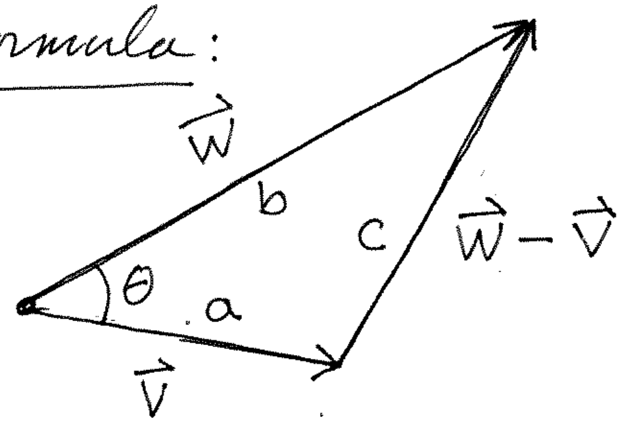
$$\vec{v}\cdot\vec{v} = |\vec{v}|^2 \qquad \vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v} \qquad (c\vec{v})\cdot\vec{w} = c(\vec{v}\cdot\vec{w})$$

Idea behind the key formula:

Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



So

$$c^2 = |\vec{w}-\vec{v}|^2 = (\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v})$$

$$= \vec{w}\cdot\vec{w} - \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{v}$$

$$= |\vec{w}|^2 + |\vec{v}|^2 - 2\,\vec{v}\cdot\vec{w} = a^2 + b^2 - 2\,\vec{v}\cdot\vec{w}$$

Solving for $\vec{v}\cdot\vec{w}$ gives

$$\vec{v}\cdot\vec{w} = \frac{1}{2}\Big(\underbrace{a^2+b^2-c^2}_{=\,2ab\cos\theta}\Big) = |\vec{v}||\vec{w}|\cos\theta$$