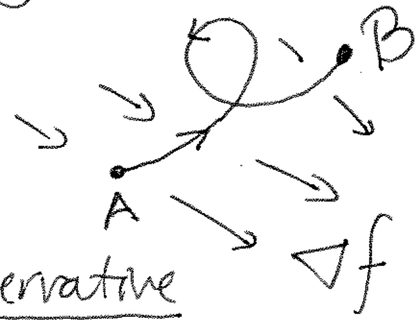
Lecture 19: Conservative Vector Fields (§16.3)

Previously on Math 241:

Fundamental Thm of Line Integrals: $f: \mathbb{R}^n \to \mathbb{R}$ differentiable and C a curve joining A to B.

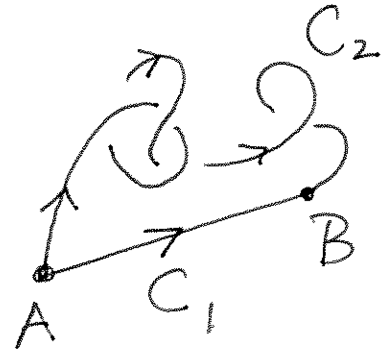
Then $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$



A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative when $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$.

By the Fund. Thm, if $\vec{F}$ is conservative then

(a) Independence of path:
If C_1 and C_2 are two paths joining A to B, then

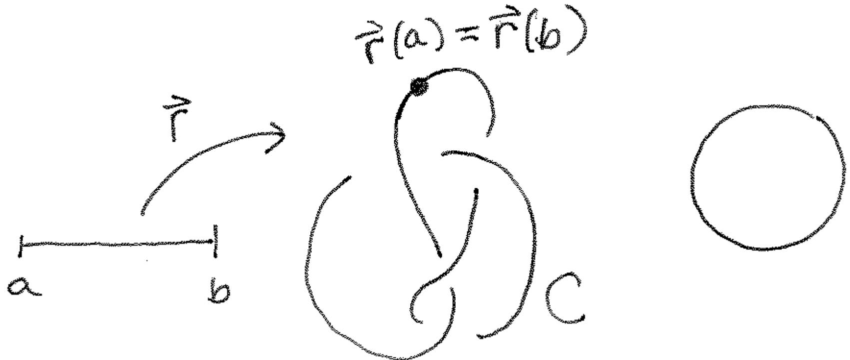


$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

as both are $f(B) - f(A)$ where $\vec{F} = \nabla f$.

(b) If C is a closed curve (i.e. starts and ends at the same point, like a circle) then

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

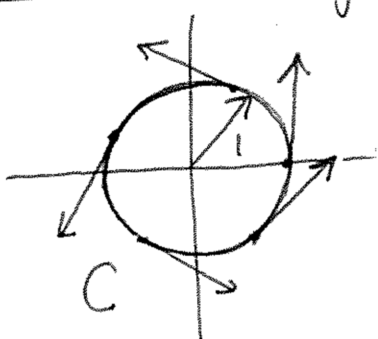


<u>Reason</u>: If $\vec{r}(a)=\vec{r}(b)$ and $\vec{F}=\nabla f$ then

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

same input

<u>Ex</u>: $\vec{F} = (-y, x)$ [From 1st lecture on vector fields.]



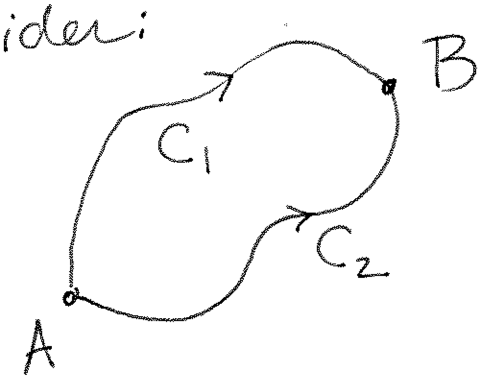
For the unit circle, $\vec{F}=\vec{T}$ and so

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}\cdot\vec{T})\,ds = \int_C ds = 2\pi$$

So by (b), $\vec{F}$ is not conservative.

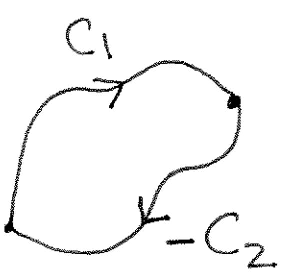
[On tomorrow's worksheet, will apply (a) in a similar way.]

<u>Note</u>: Conditions (a) and (b) are equivalent. For example, if (b) holds for $\vec{F}$ consider:



Let $-C_2$ be C_2 backwards.

Take C:



Then $0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r}$

$$= \int_{C_1} \vec{F}\cdot dr - \int_{C_2} \vec{F}\cdot d\vec{r} \implies \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

[Conversely, if (a) holds for $\vec{F}$, then break a closed curve into 2 segments to see that $\int_C \vec{F}\cdot ds = 0$.]

<u>Note</u>: Being conservative is subtle; for example is $\vec{F} = (y, x)$ conservative? [Compare with above ex.]

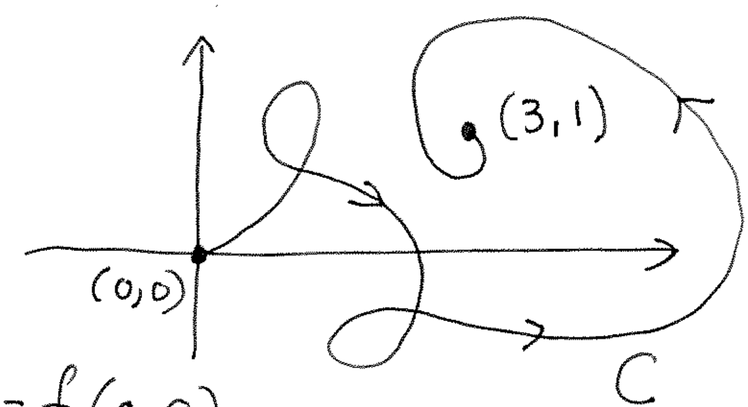
<u>A</u>. Yes. Need $f(x,y)$ with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$.

First condition gives $f = \int y\, dx = xy + C(y)$

treat as constant

Second is similar, so take $f = xy$.

<u>Usefullness is clear</u>:



$$\int_C \vec{F}\cdot d\vec{r} = f(3,1) - f(0,0)$$
$$= 3\cdot 1 - 0 = 3.$$

[Will give two tests for when $\vec{F}$ is conservative, but first need to introduce some terms...]

Focus on a vector field $\vec{F}$ on $\mathbb{R}^2$, with domain D.

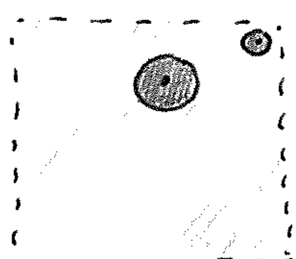
Ex: $\vec{F} = (x, y) \quad D = \mathbb{R}^2$

$$\vec{F} = \frac{1}{x^2+y^2}(-y, x) \quad D = \{(x,y) \neq (0,0)\}$$

Properties D may have:

Open: Roughly, D contains none of its boundary points. Precisely, each point of D is the center of a disc that is also contained in D.

Ex: $D = \left\{ (x,y) \;\middle|\; \begin{array}{l} 0 < x < 1 \\ 0 < y < 1 \end{array} \right\}$



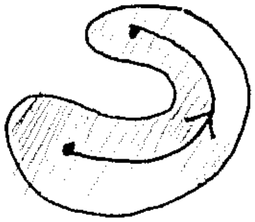
$D = \mathbb{R}^2$

$D = \{(x,y) \neq \{0,0\}\}$

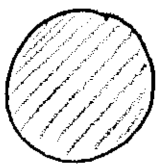
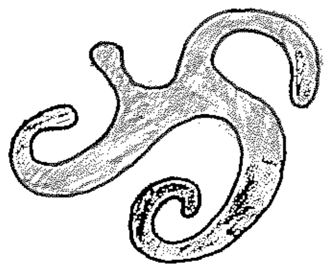
Closed: Opposite of open.

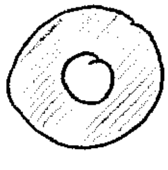
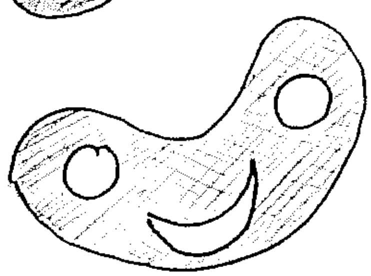
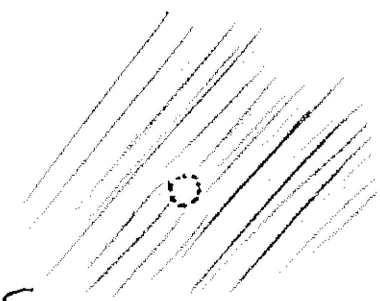
[Precisely, D is closed exactly when its complement is open.]

Connected: Any two points in D can be joined by a path inside D.

Ex:    **Non Ex**: 

Simply Connected: Connected + no holes

Ex:    **Ex**: 

Non Exs:   

$\{(x,y) \neq (0,0)\}$

Thm A: Suppose $\vec{F}$ is a vector field on an open connected set D in $\mathbb{R}^2$. Then $\vec{F}$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Thm B: Suppose $\vec{F} = (P, Q)$ is a vector field on an open *simply* connected set D in $\mathbb{R}^2$. Then $\vec{F}$ is conservative if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ on D.

[Thm A is nice but hard to apply. Will talk about next time.]

Reason for Thm B: Suppose $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$$\underbrace{\tfrac{\partial f}{\partial x}}_{P} \quad \underbrace{\tfrac{\partial f}{\partial y}}_{Q}$$

Then
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$
and
$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

Point: mixed partials are equal!

[We'll come back to why $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ conservative in about a month.]

Ex: $\vec{F} = (y, x)$ Then $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow$ conservative

$\vec{F} = (-y, x)$ Then $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial Q}{\partial x} \Rightarrow$ not conser.

Ex: $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ on $D = \{(x,y) \neq 0\}$

On HW: Check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ but its not path independent, and hence not conservative.

Point: D is not simply connected.