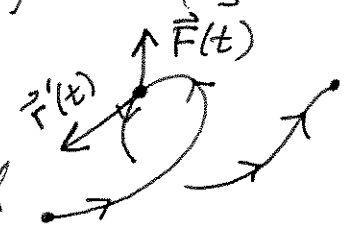


Lecture 18: More on integrating vector fields (§16.2)

①

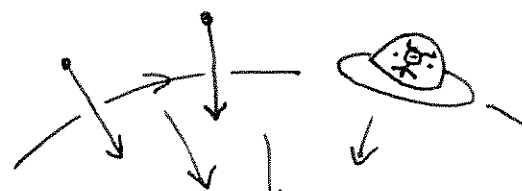
Last time:  $C$  oriented curve in  $\mathbb{R}^n$   
 $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$  vector field



$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

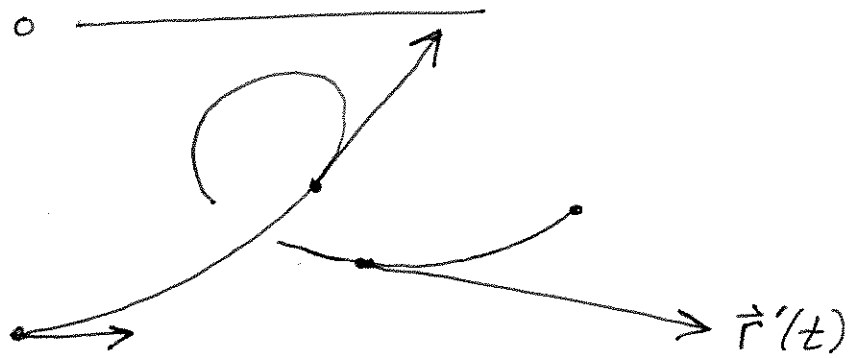
for any parameterization  $\vec{r}: [a, b] \rightarrow \mathbb{R}^n$  of  $C$ .

Ex: Work =  $\int_C \vec{F} \cdot d\vec{r}$



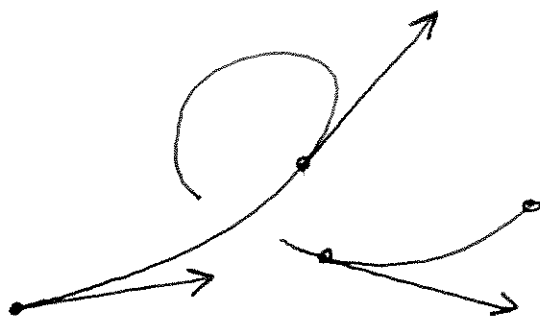
[Other examples from E+M, see HW.]

Alternate viewpoint

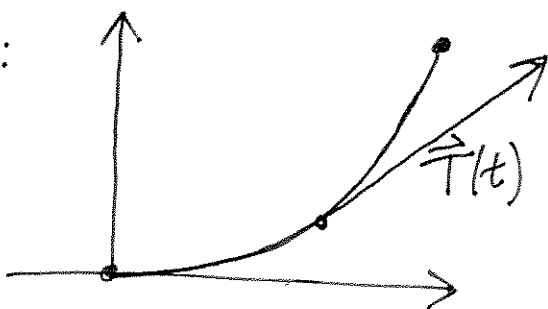


Unit tangent vectors

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$



Ex:



$$\vec{r}(t) = (t, t^2) \quad \vec{r}'(t) = (1, 2t)$$

$$|\vec{r}'(t)| = \sqrt{1+4t^2}$$

$$\vec{T}(t) = \left( \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right)$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = |\vec{r}'(t)| \vec{T}(t) \quad (2)$$

$$= \int_a^b \left( \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \right) \underbrace{|\vec{r}'(t)|}_{ds} dt$$

$$= \int_C \vec{F} \cdot \vec{T} ds \quad \left[ \begin{array}{l} \text{Integral of a fn with} \\ \text{respect to arc length} \end{array} \right]$$

Note that  $\int$  doesn't change if we use a different parameterization, unless we travel the other way along  $C$  and then  $\vec{T}(t) \leftrightarrow -\vec{T}(t)$ .

Thus if  $-C = C$  oriented the other way

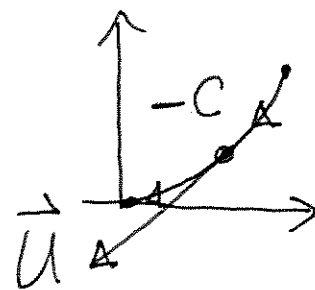
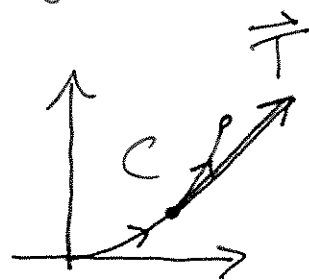
have:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$$

Integrals of functions don't depend on orient

$$= \int_{-C} \vec{F} \cdot \vec{T} ds$$

$$= - \int_{-C} \vec{F} \cdot \vec{U} ds = - \int_{-C} \vec{F} \cdot d\vec{r}$$



[So switching orientation of  $C$  changes the sign of  $\int_C \vec{F} \cdot d\vec{r}$ ]

③

Alternate notation:  $C$  curve in  $\mathbb{R}^2$ , parameterized by  $\vec{r}: [a, b] \rightarrow \mathbb{R}^2$ , and  $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a vector field

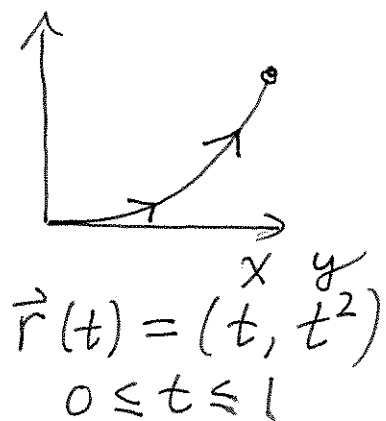
$$\vec{r}(t) = (x(t), y(t)) = x(t)\vec{i} + y(t)\vec{j}$$

$$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$$

Then

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt \\ &= \int_a^b (P(\vec{r}(t))x'(t) + Q(\vec{r}(t))y'(t)) dt \\ &= \int_a^b P(\vec{r}(t)) \underbrace{x'(t) dt}_{dx} + \int_a^b Q(\vec{r}(t)) \underbrace{y'(t) dt}_{dy} \\ &= \int_C P dx + \int_C Q dy \quad [\text{New notation.}] \end{aligned}$$

Ex:



Evaluate:  $\int_C y dx + dy$

$$= \int_0^1 t^2 \underbrace{(1 dt)}_{x'(t) dt} + 1 \underbrace{(2t) dt}_{y'(t) dt}$$

$$= \int_0^1 t^2 + 2t dt = \left. \frac{t^3}{3} + t^2 \right|_{t=0}^1 = 4/3.$$

(4)

[Same as at end of last time.]

Similarly in  $\mathbb{R}^3$ :  $\int_C P dx + Q dy + R dz$

is just  $\int_C \vec{F} \cdot d\vec{r}$  where  $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

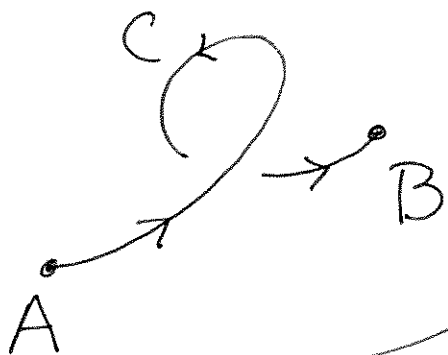
Fundamental Theorem of Calculus:  $f: [a, b] \rightarrow \mathbb{R}$

which is differentiable. Then  $\int_a^b f'(t) dt = f(b) - f(a)$ .

Fundamental Theorem of Line Integrals: Suppose

$f: \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable and  $C$  a curve in  $\mathbb{R}^n$

from  $A$  to  $B$ . Then  $\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$



Reason: Pick  $\vec{r}: [a, b] \rightarrow C$   
where  $\vec{r}(a) = A$  and  $\vec{r}(b) = B$ .

Then 
$$\int_C \nabla f \cdot dr = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

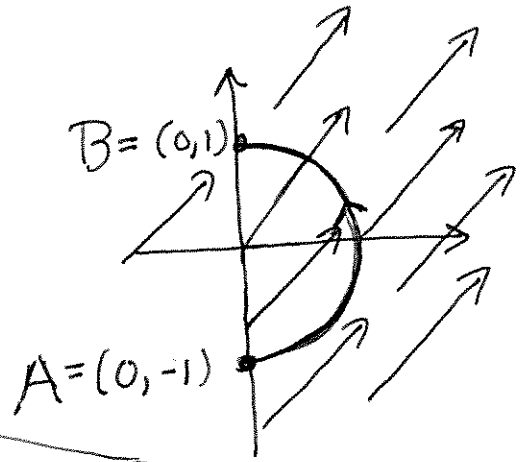
$$= \int_a^b \underbrace{(D_{\vec{r}'(t)} f)(\vec{r}(t))}_{\text{rate } f \text{ changes as we move along curve at time } t} dt \quad \nabla f(\vec{r}(t)) \quad \vec{r}'(t) \quad (5)$$

rate  $f$  changes as we move along curve at time  $t$

$$= \int_a^b (\text{rate of change in } f(t)) dt = f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(B) - f(A).$$

Ex:  $f(x,y) = x+y$      $\vec{F} = \nabla f = (1,1)$



$$\int_C \vec{F} \cdot d\vec{r} = \int_{-\pi/2}^{\pi/2} (1,1) \cdot \underbrace{(-\sin t, \cos t)}_{\vec{r}'(t)} dt \quad \left. \begin{array}{l} \text{Curve } C. \\ \vec{r}(t) = (\cos t, \sin t) \\ -\pi/2 \leq t \leq \pi/2 \end{array} \right\}$$

$$= \int_{-\pi/2}^{\pi/2} -\sin t + \cos t dt$$

$$= \cos t + \sin t \Big|_{t=-\pi/2}^{\pi/2} = 1 - (-1) = 2.$$

Fund. Thm. says as  $\vec{F} = \nabla f$

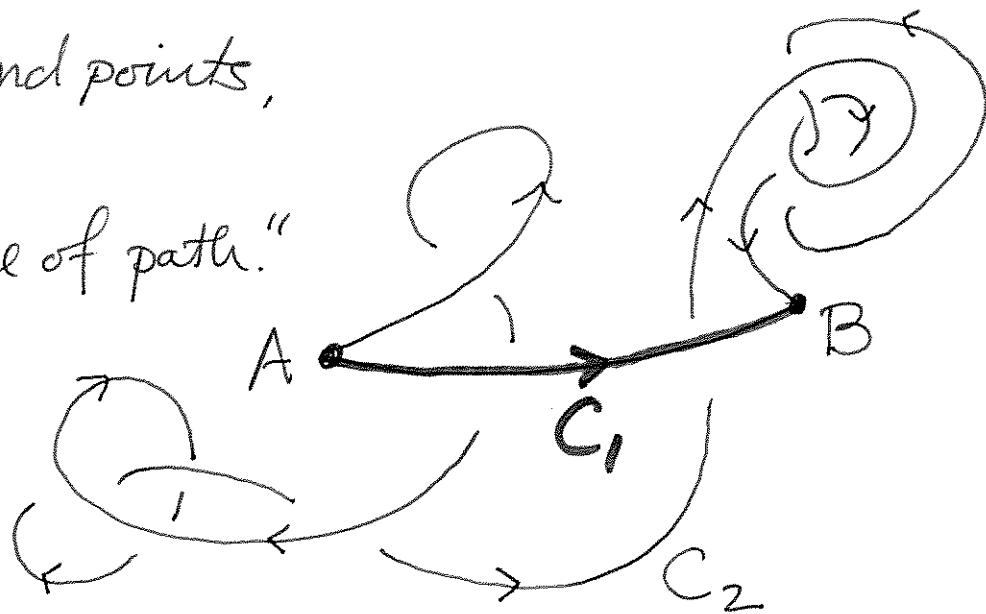
$$\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = 1 - (-1) = 2 \quad \checkmark$$

Consequence: If  $\vec{F}$  is conservative ( $= \nabla f$ ) (6)

then  $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$  if  $C_1$  and  $C_2$  have

the same endpoints,

"Independence of path."



[ Next two lectures will explore conservative vector fields and characterize them by properties of their path integrals. ]