Lecture 17: Vector fields (§16.1 and 16.2)

Last time:

C curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

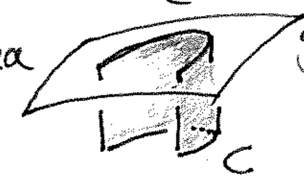
where $\vec{r}: [a,b] \to \mathbb{R}^2$ is a parameterization of C

Uses:

① Average of f on C $= \dfrac{1}{\text{Len}(C)} \displaystyle\int_C f\,ds$

② Total mass $= \displaystyle\int_C f\,ds$ ← density fn.

③ Area



Note: $ds = |\vec{r}\,'(t)|\,dt$ is the arc-length element

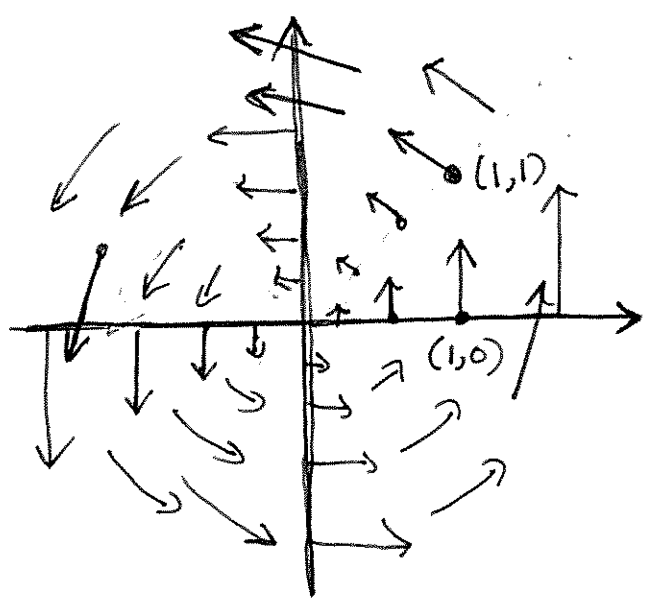
Also $\displaystyle\int_C 1\,ds = \text{Length}(C)$

[Everything works for curves in $\mathbb{R}^3$ except use ③.
Will learn more about diff param. of the same C in section.]

Vector fields (§16.1 and 16.2)

For $\mathbb{R}^2$, a vector field is a function $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

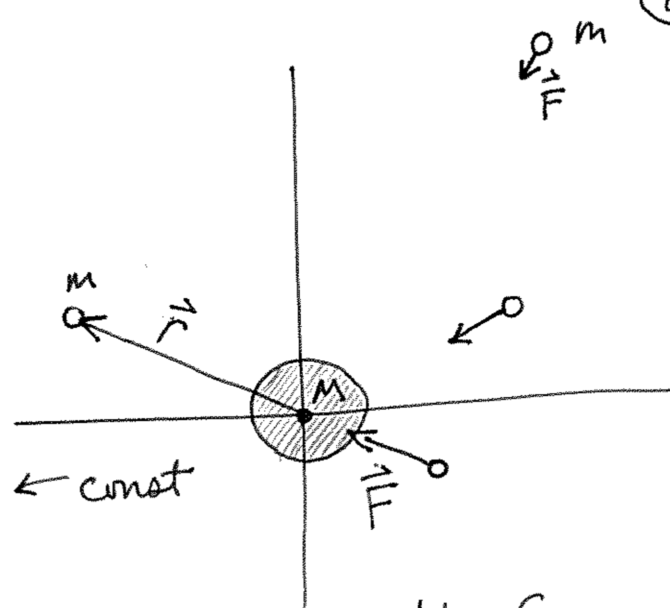
Ex: $\vec{F}(x,y) = -\dfrac{y}{2}\vec{i} + \dfrac{x}{2}\vec{j}$



Uses:
- Wind speed/direction
- Fluid flow
- Force magnitude/direction
- Electric/magnetic fields

Ex: Gravity

Large mass M at $(0,0)$.
Force $\vec{F}$ on small mass m
depends on position $\vec{r} = (x,y)$
and points in direction $-\vec{r}$.



Newton's Law: $|\vec{F}| = \dfrac{MmG}{|\vec{r}|^2}$ ← const

As $\vec{F} = -c\vec{r}$, have $|\vec{F}| = c|\vec{r}| \Rightarrow c = \dfrac{MmG}{|\vec{r}|^3}$

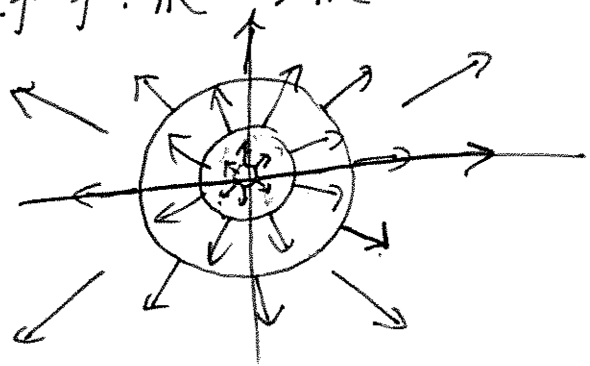
So $\vec{F} = -\dfrac{MmG}{|\vec{r}|^3}\vec{r}$

[For several bodies, add vector fields. An electric field generated by a charged partical is similar.]

[Where have we seen vector fields before in this class?]

Ex: Gradients: If $f: \mathbb{R}^2 \to \mathbb{R}$ then $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$

① $f(x,y) = x^2+y^2$
$\nabla f = (2x, 2y)$



② $f(x,y) = \dfrac{MmG}{\sqrt{x^2+y^2}} = \dfrac{MmG}{|\vec{r}|}$. Then

$$\nabla f = MmG\left(-\tfrac{1}{2}(x^2+y^2)^{-3/2}\cdot(2x),\ \_\_\_\_\right)$$

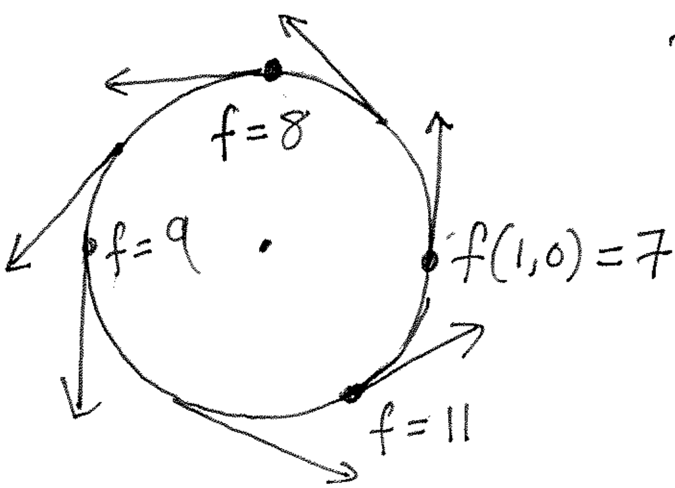
$$= -MmG\left(\frac{x}{(x^2+y^2)^{3/2}},\ \frac{y}{(x^2+y^2)^{3/2}}\right)$$

$$= \frac{-MmG}{|\vec{r}|^3}\vec{r} = \vec{F}$$

In general, when $\vec{F} = \nabla f$ we call f a <u>potential function</u> for $\vec{F}$ and say $\vec{F}$ is a <u>conservative vector field</u>. [Think potential energy / conservation of energy.]

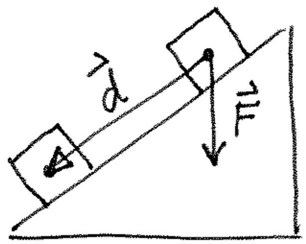
<u>Q</u>: Is $\vec{F} = -\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}$ conservative?

<u>A</u>: No. Suppose $\vec{F} = \nabla f$. Since $\vec{F}$ is tangent to the unit circle, following the circle increases f.



But going all the way around, we end up back at $(1,0)$.

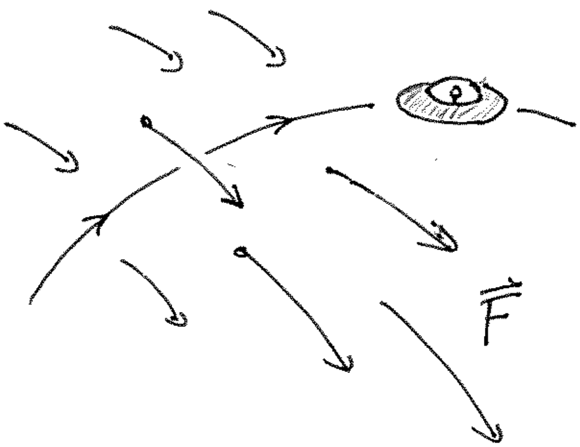
Integrating Vector Fields (§16.2)



Work done by gravity: $W = \vec{F} \cdot \vec{d}$

[Assumes const force.]

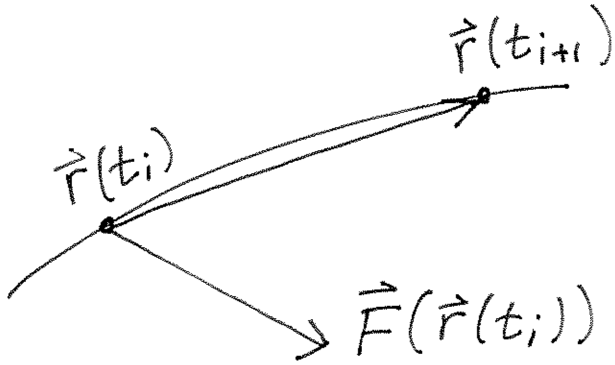
How much work does gravity do here?



Motion of ship $\vec{r}: \mathbb{R} \to \mathbb{R}^2$

Force of gravity: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

Break into segments:



$$\text{Work done here} \approx \vec{F}(\vec{r}(t_i)) \cdot \underbrace{\left(\vec{r}(t_{i+1}) - \vec{r}(t_i)\right)}_{\approx \Delta t\, \vec{r}\,'(t_i)}$$

$$\approx \left(\vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\right) \Delta t$$

Sum up and take $\Delta t \to 0$ to learn

$$\text{Total Work} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

<u>General Setup:</u> C curve in $\mathbb{R}^n$

$\vec{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a vector field

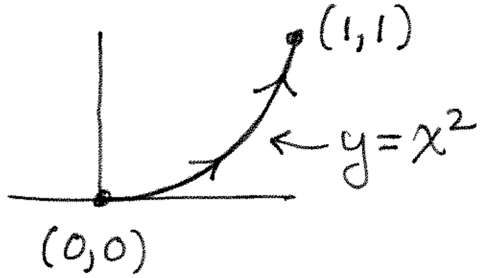
Set

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

for any parameterization $\vec{r} : [a,b] \longrightarrow \mathbb{R}^n$ of C.

[Note: Answer only depends on direction of param.]

<u>Ex</u>: C



$\vec{r}(t) = (t, t^2)$ for $0 \leq t \leq 1$

$\vec{r}\,'(t) = (1, 2t)$

For $\vec{F} = (y, 1)$ we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^2, 1) \cdot (1, 2t)\, dt$$

$$= \int_0^1 t^2 + 2t \, dt = \frac{t^3}{3} + t^2 \Big|_{t=0}^{t=1} = \frac{4}{3}$$

Here's why this is consistent with the work interpretation of $\int_C \vec{F} \cdot d\vec{r}$