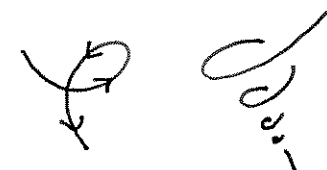
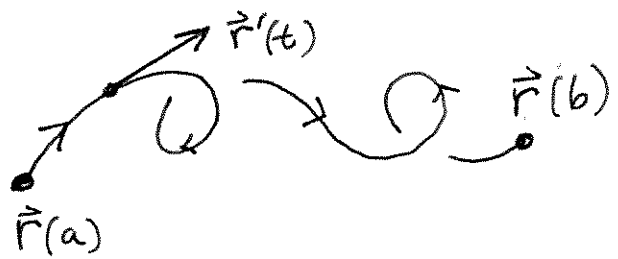


Lecture 16: Integration along curves (§13.3 and 16.2)

Last time: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2$ or \mathbb{R}^3



$$\text{Length} = \int_a^b |\vec{r}'(t)| dt$$



Really two concepts:

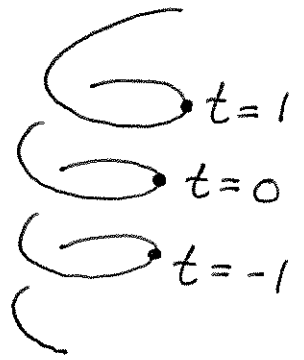
Curve: A set of points in \mathbb{R}^3 looking like:



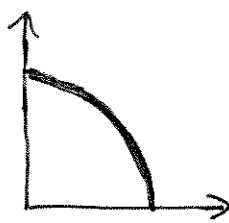
Parameterization:

$$\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$$

Instructions for moving along a curve.




[Any curve has many parameterizations.]

Ex:  $C = 1/4$ of a unit circle in \mathbb{R}^2 .

① $\vec{r}: [0, \pi/2] \rightarrow \mathbb{R}^2$ $\vec{r}(t) = (\cos t, \sin t)$

$$\vec{r}'(t) = (-\sin t, \cos t)$$

[Parameterizing by angle 

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$$

$$\text{Length} = \int_0^{\pi/2} |\vec{r}'(t)| dt = \int_0^{\pi/2} 1 dt = \pi/2.$$

⑥ $\vec{r}: [0, 1] \rightarrow \mathbb{R}^2 \quad \vec{r}(t) = (t, \sqrt{1-t^2})$ ②

[Parameterizing by x]

$$\vec{r}'(t) = (1, \frac{1}{2}(1-t^2)^{-1/2} \cdot (-2t))$$

$$= (1, \frac{-t}{\sqrt{1-t^2}})$$

$$|\vec{r}'(t)| = \sqrt{1^2 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

$$\text{Length} = \int_0^1 \frac{1}{\sqrt{1-t^2}} dt = \arcsin(t) \Big|_{t=0}^1 = \pi/2 - 0 = \boxed{\pi/2} \checkmark$$

Integrating a function on a curve (§ 16.2)

C curve in $\mathbb{R}^2 \quad f: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt$$

where $\vec{r}: [a, b] \rightarrow \mathbb{R}^2$ is a parameterization of C .

[Turns out not to depend on \vec{r} , just C .]

Some meanings:

① $f = \text{temperature}$

Compare: Average of $f: \mathbb{R} \rightarrow \mathbb{R}$ on $[a, b]$ is $\frac{1}{b-a} \int_a^b f(x) dx$

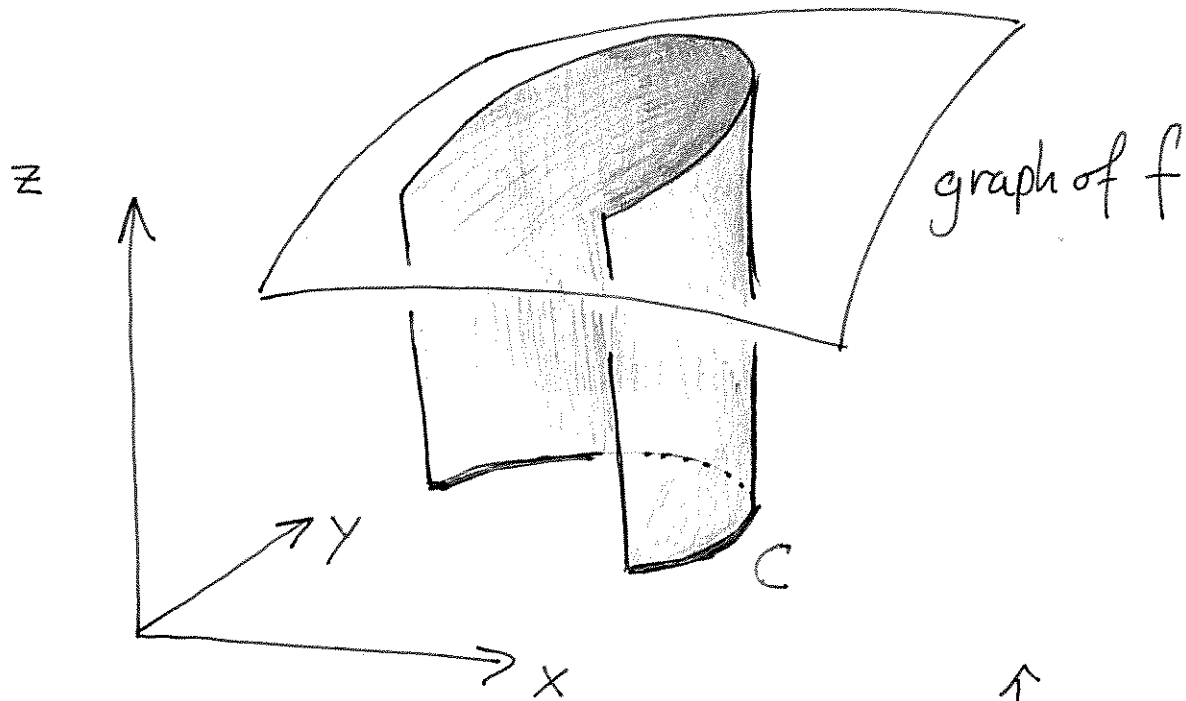
$$\text{Average temperature along } C = \frac{1}{\text{Length}(C)} \int_C f ds$$

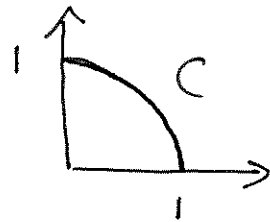
② $f =$ density of material (mass/length)
curve is made of

③

$$\text{Mass of curve} = \int_C f ds$$

③ Area of region above C and below the graph of f .



Ex: Find the average of $f(x,y) = x$ on 

$$\vec{r}(t) = (\cos t, \sin t) \quad 0 \leq t \leq \pi/2$$

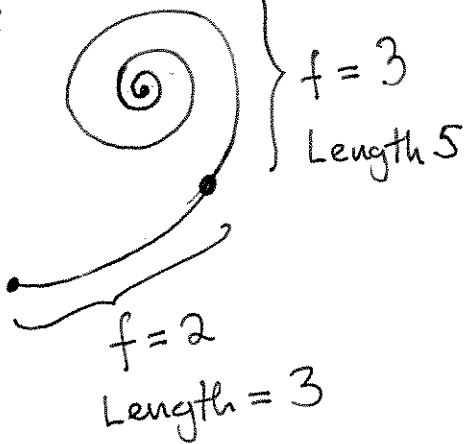
$$\int_C f ds = \int_0^{\pi/2} f(\vec{r}(t)) |\vec{r}'(t)| ds = \int_0^{\pi/2} \cos t \cdot 1 dt$$

$$= \sin t \Big|_0^{\pi/2} = \sin(\pi/2) - \sin(0) = 1 - 0 = 1$$

$$\text{Average} = \frac{1}{\text{Length}} \int_C f ds = \frac{2}{\pi} \approx 0.6366$$

Understanding these integrals:

Ex:

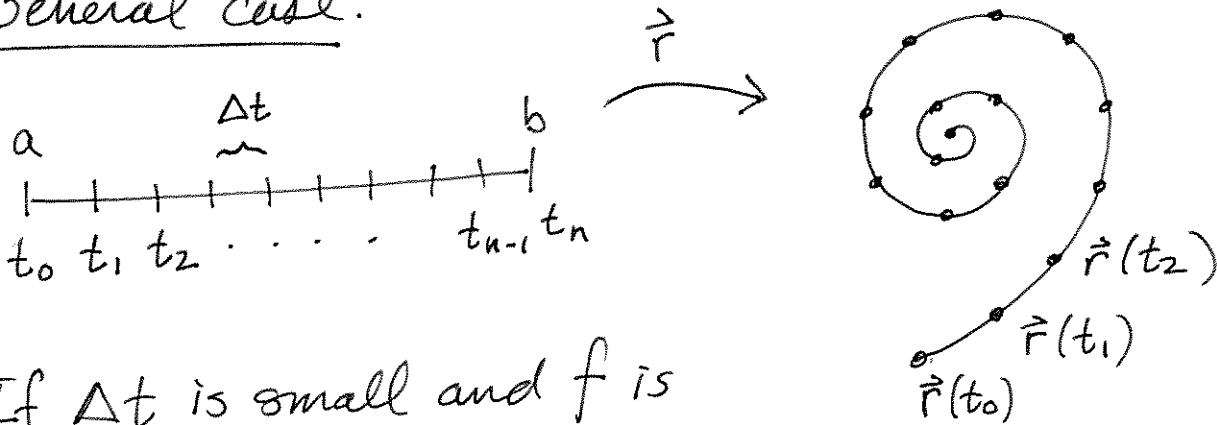


[What is the average here? First guess of 2.5 is too small...]

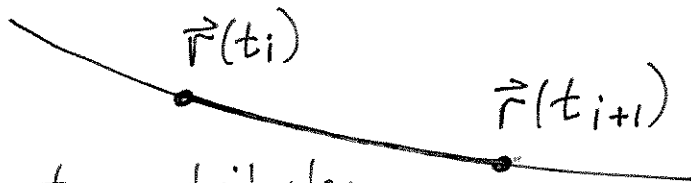
Average of f on C

$$\begin{aligned}
 &= \left(\begin{array}{c} \text{portion} \\ \text{where} \\ f=2 \end{array} \right) \cdot 2 + \left(\begin{array}{c} \text{portion} \\ \text{where} \\ f=3 \end{array} \right) \cdot 3 \\
 &= \left(\frac{3}{8} \right) \cdot 2 + \left(\frac{5}{8} \right) \cdot 3 \\
 &= 21/8 = 2\frac{5}{8}
 \end{aligned}$$

General case:



If Δt is small and f is continuous, then f is almost constant on



So this segment contributes

$$\approx \frac{(\text{length of seg.})}{(\text{length of } C)} f(\vec{r}(t_i))$$

to the average.

(5)

Now the segment has length $\approx |\vec{r}'(t_i)| \Delta t$
by linear approximation, and so

$$\text{Average} \approx \sum_{i=0}^{n-1} \frac{(\text{len of } i^{\text{th}} \text{ seg})}{(\text{len of } C)} f(\vec{r}(t_i))$$

$$\approx \frac{1}{\text{len}(C)} \sum_{i=0}^{n-1} f(\vec{r}(t_i)) |\vec{r}'(t_i)| \Delta t$$

As $\Delta t \rightarrow 0$, we get

$$\begin{aligned} \text{Average} &= \frac{1}{\text{len}(C)} \int_a^b f(\vec{r}(t)) \underbrace{|\vec{r}'(t)|}_{\substack{\uparrow \\ \text{Arc length} \\ \text{element}}} dt \\ &= \frac{1}{\text{len}(C)} \int_C f ds \end{aligned}$$

Compare:

$$\text{Average of } f \text{ on } [a, b] = \frac{1}{b-a} \int_a^b f(t) dt$$

