

Lecture 15: Introduction to space curves (§13.1-4)

Next two weeks: ~~Sections 13.1-13.4~~ "Curves in space..."

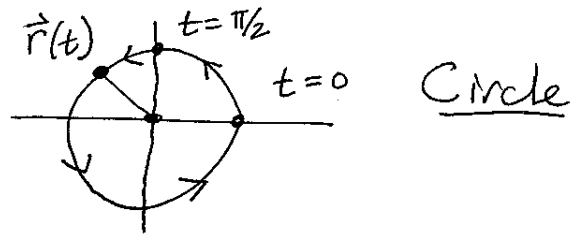
Chapter 13, Section 16.1-16.3

Last time: Not really relevant.

Curve: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2$ or \mathbb{R}^3 (vector-valued function)

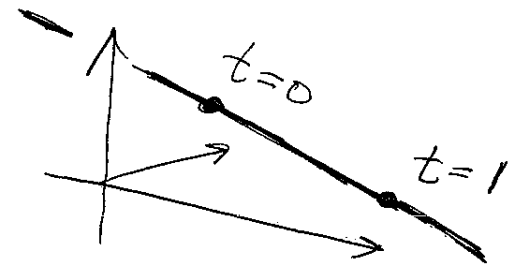
Ex: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^2$

$$\vec{r}(t) = (\cos t, \sin t)$$



Ex: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$

$$\begin{aligned} \vec{r}(t) &= (1, 0, 1) + t(2, 1, -1) \\ &= (1+2t, t, 1-t) \end{aligned}$$

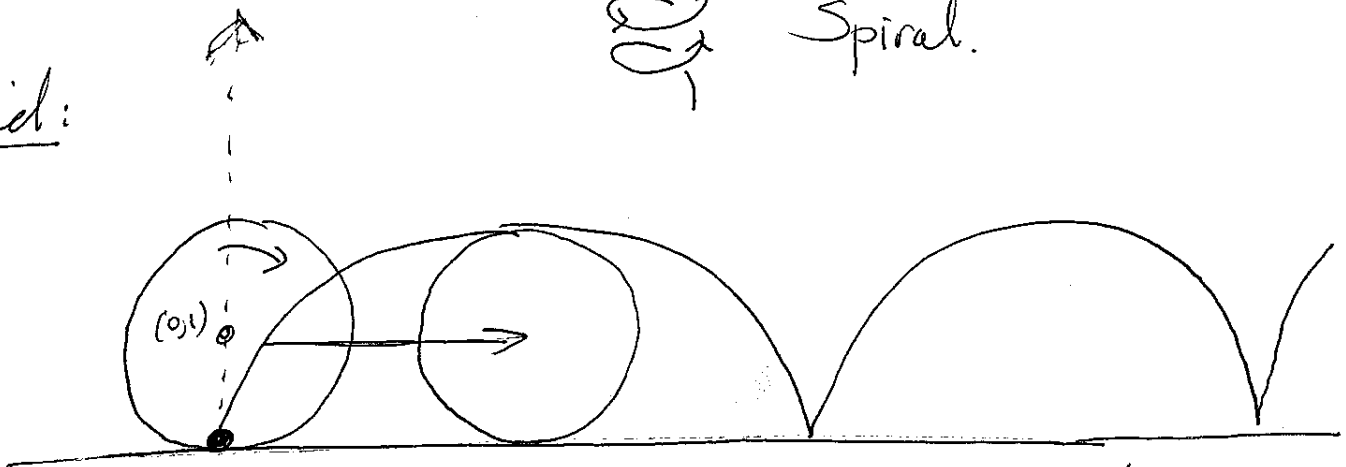


Ex: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$

$$\vec{r}(t) = (\cos t, \sin t, t)$$



Cycloid:

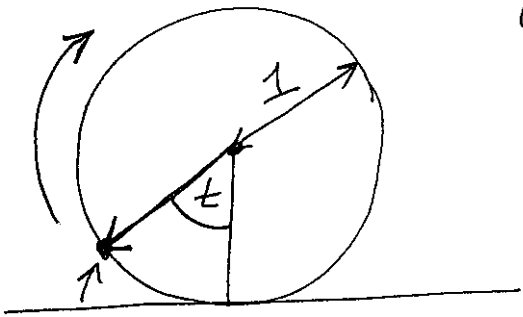


Suppose a wheel with radius 1 rotates at 1 radian/see

Thus after time t , the center has moved a distance t . So if center is at $(0,1)$ at $t=0$, its at $(t,1)$. Combined with the rotation,

we get

$$\vec{r}(t) = (t, 1) + (-\sin t, -\cos t) = (t - \sin t, -\cos t)$$

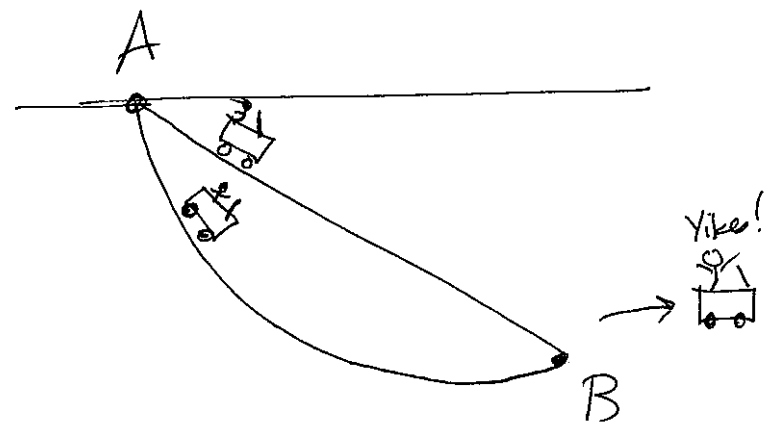
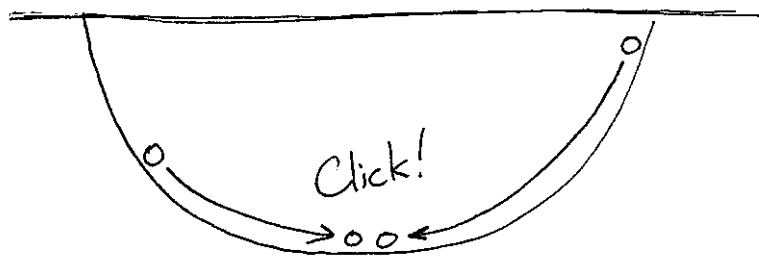


vector is $(-\sin t, -\cos t)$

Upside down, this curve has the following remarkable properties:

Tautochrone

Brachistochrone

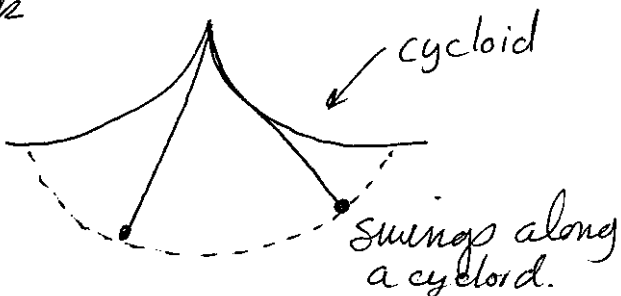


Same time to bottom

minimizes time from A to B.

Huygens (1650s) pendulum clock

This is an infinite dimensional min/max problem.



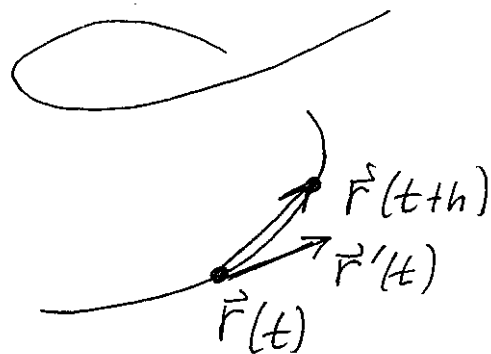
("Calculus of Variations")

Derivatives: $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$ $\vec{r}(t) = (r_1(t), r_2(t), r_3(t))$ 53

$$\vec{r}'(t) = (r_1'(t), r_2'(t), r_3'(t))$$

$$= \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

↑ component fns.



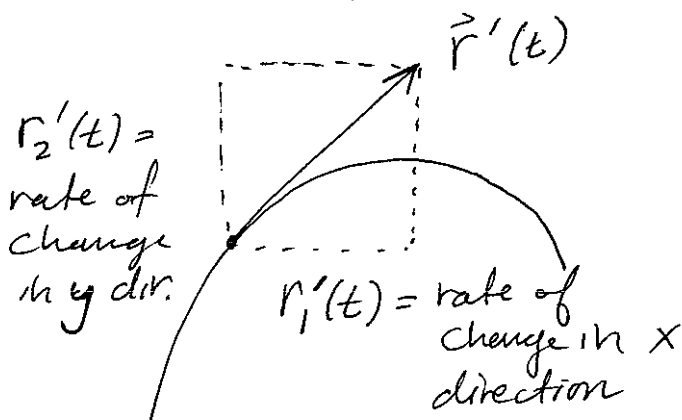
The vector $\vec{r}'(t)$ is tangent to the curve at $\vec{r}(t)$. Geometrically, it is the velocity vector of the motion at time t .

(notice the units of $r'(t)$ are $\frac{\text{distance}}{\text{time}}$). The

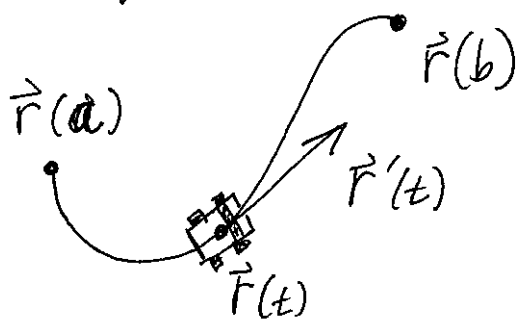
Speed of the motion at time t is $|\vec{r}'(t)|$

Also $\vec{r}': \mathbb{R} \rightarrow \mathbb{R}^3$ and

$\vec{r}''(t)$ is the acceleration at time t .



Length of a curve: $\vec{r} : [a, b] \rightarrow \mathbb{R}^2$ (or \mathbb{R}^3)

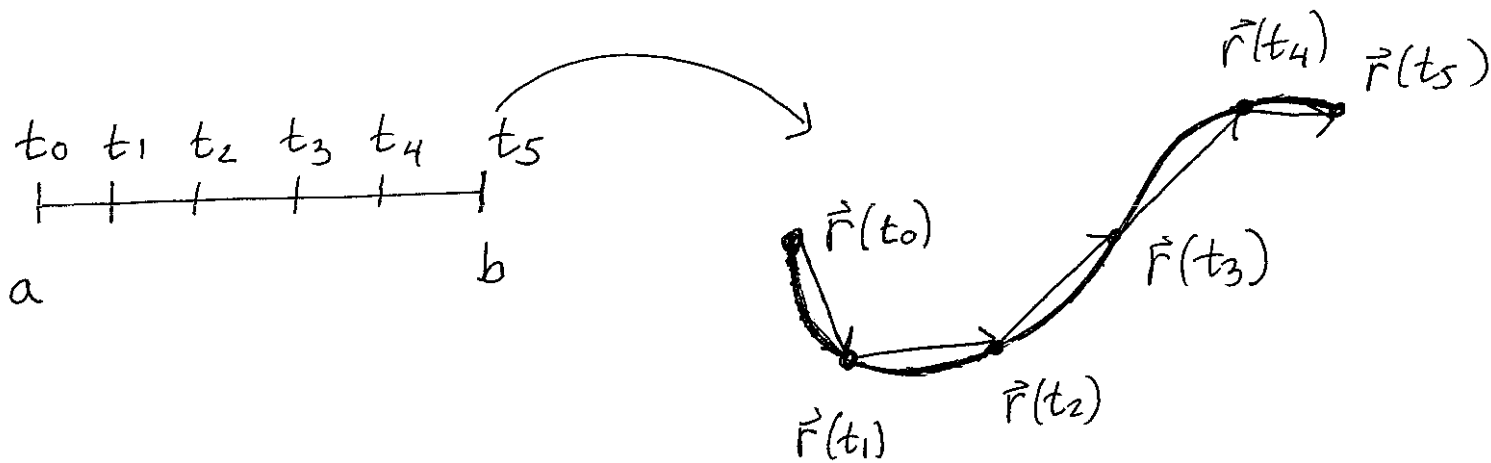


How long is this path?

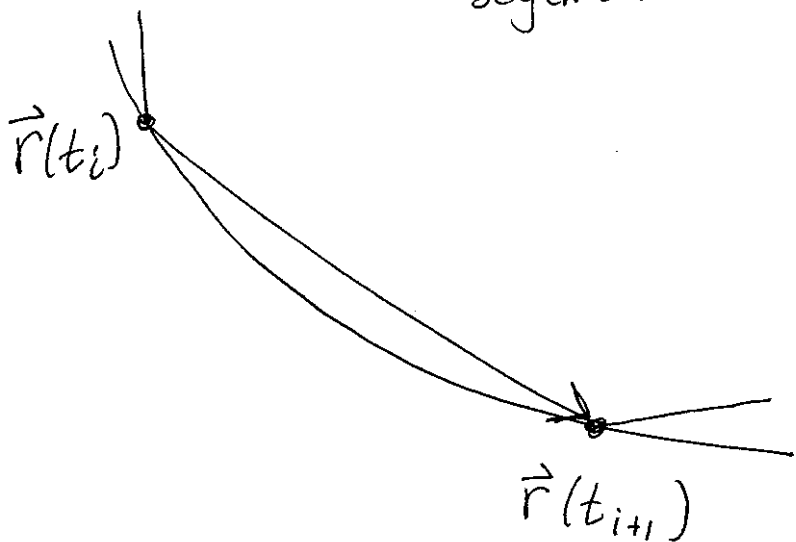
If we view this the motion of something, have:

$$\text{distance traveled} = \int_a^b \text{speed at time } t \, dt = \int_a^b |\vec{r}'(t)| \, dt$$

Another point of view: Approximate by straight segments



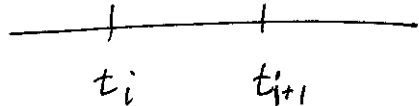
$$\text{Length} \approx \sum \text{length of segments} = \sum_{i=0}^n |\vec{r}(t_{i+1}) - \vec{r}(t_i)|$$



Linear approximation: (just like in one var.)

$$\vec{r}(t_i + \Delta t) \approx \vec{r}(t_i) + \vec{r}'(t_i) \Delta t$$

Take Δt and so



$$|\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx |\vec{r}'(t_i)| \Delta t.$$

So

$$\text{length} \approx \sum_{i=0}^n |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx \underbrace{\sum_{i=0}^n |\vec{r}'(t_i)| \Delta t}_{\text{Riemann Sum}}$$

As $\Delta t \rightarrow 0$, these

Riemann sums converge to

$$\int_a^b |\vec{r}'(t)| dt, \text{ and also to the length.}$$

