Lecture 14: Constrained min/max (§14.8)

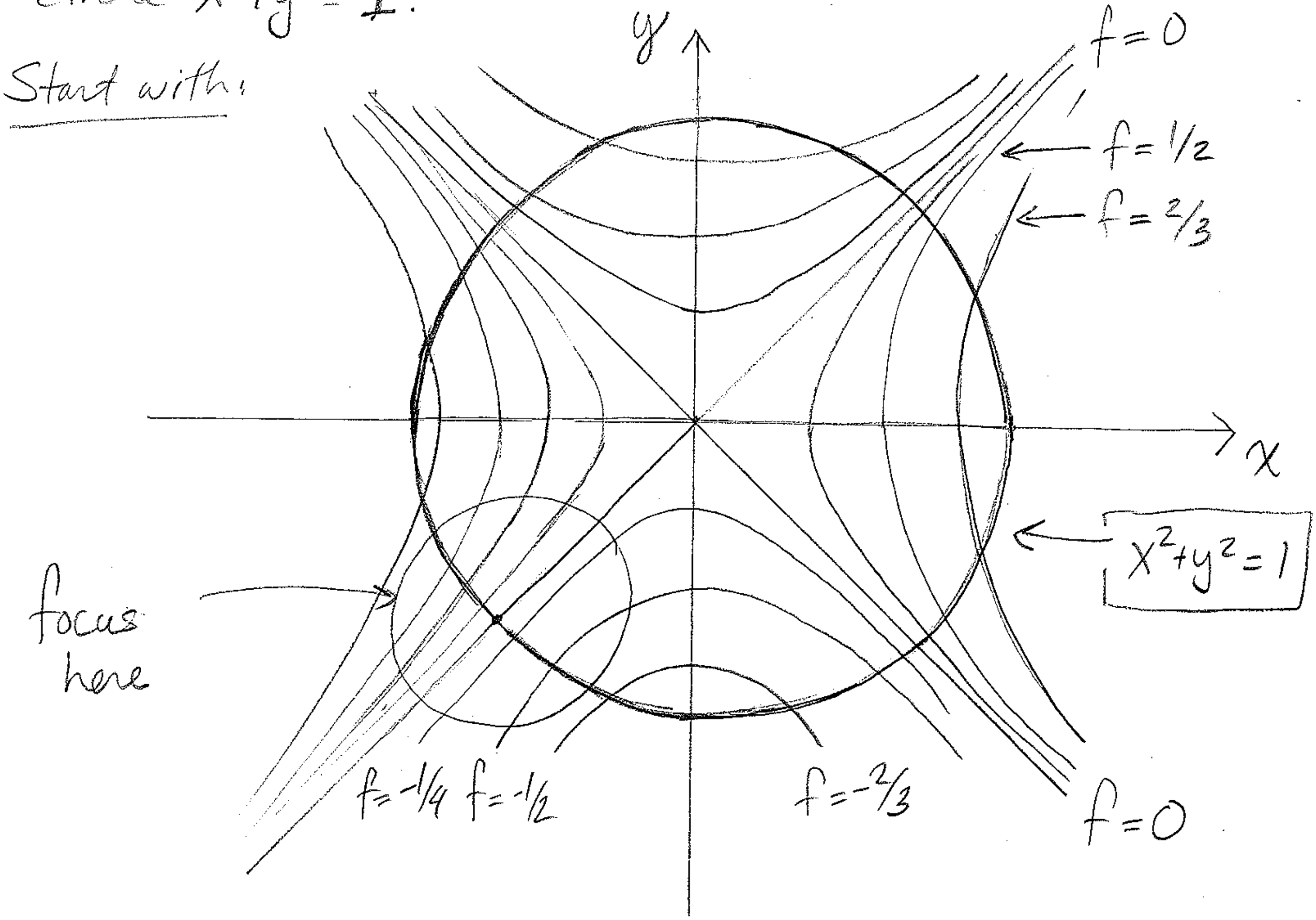
Last time:

Extreme Value Thm: f continuous on D in $\mathbb{R}^n$.
If D is closed and bounded, then f has both an absolute min and an absolute max on D.
These occur at critical pts of f or on the boundary of D.

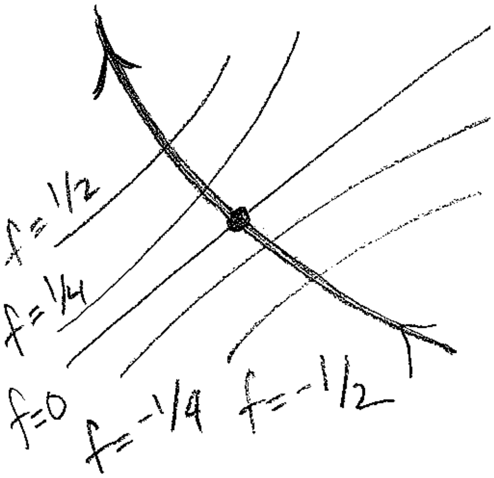
↑ focus today

Ex: Find the max of $f(x,y) = x^2 - y^2$ on the unit circle $x^2+y^2=1$.

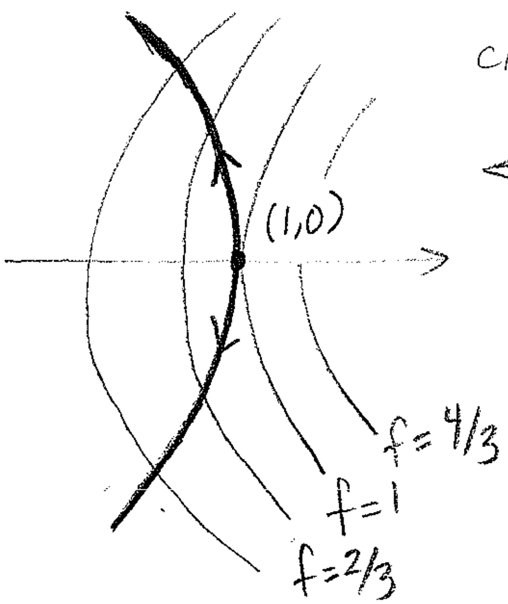
Start with:



When we have this picture, don't have a local max since can increase f by moving clockwise along the circle.



However, when the level set of f is tangent to the circle, we can have a local max.



← Here, f decreases as we move away from $(1,0)$ in either direction.

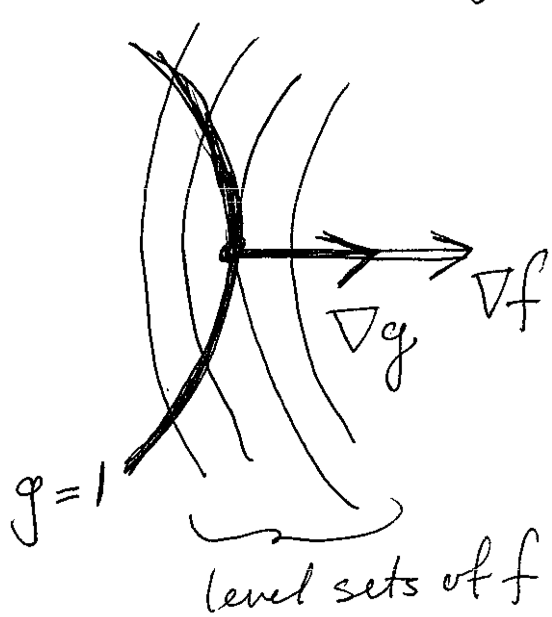
Four tangencies in this example:

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
Value of f	1	1	-1	-1
Type	max	max	min	min

absolute in each case; the circle is closed and bounded.

Finding these tangencies in general:

View the circle as the level set $g(x,y) = 1$ where $g(x,y) = x^2+y^2$. At a tangency, ∇f is at right angles to the circle:



Key: At a tangency, ∇f and ∇g point in the same direction:

$$\nabla f = \lambda \nabla g$$

λ ← some number.

$\boxed{\text{Lagrange Multipliers discovered by Euler}}$

Ex: $g(x,y) = x^2+y^2 = 1$

$$\nabla f = (2x, -2y) = \lambda \nabla g = \lambda (2x, 2y) = (2\lambda x, 2\lambda y)$$

So $2x = 2\lambda x$ and $-2y = 2\lambda y$.

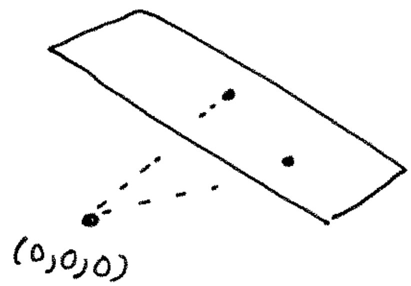
If $x \neq 0$, then $\lambda = 1$ and $y = 0 \Rightarrow x = \pm 1$.

If $y \neq 0$, then $\lambda = -1$ and $x = 0 \Rightarrow y = \pm 1$

from $x^2+y^2 = 1$.

So the critical points are $(1,0)$, $(-1,0)$, $(0,1)$, $(0,-1)$ just as before.

Ex: Find the distance from $\overbrace{x-y+2z}^{g(x,y,z)} = 6$ to $(0,0,0)$.



Minimize: $f(x,y,z) = x^2+y^2+z^2$

Subject to: $g(x,y,z) = 3$

Critical Points:

$$\nabla f = (2x, 2y, 2z) = \lambda \nabla g = \lambda(1,-1,2)$$
$$= (\lambda, -\lambda, 2\lambda)$$

Solve: $\left.\begin{aligned} 2x &= \lambda \\ 2y &= -\lambda \\ 2z &= 2\lambda \end{aligned}\right\} \Rightarrow \begin{aligned} 2y &= -\lambda = -2x \Rightarrow y = -x \\ 2z &= 2\lambda = 4x \Rightarrow z = 2x \end{aligned}$

Combined with $g = 6$, this gives

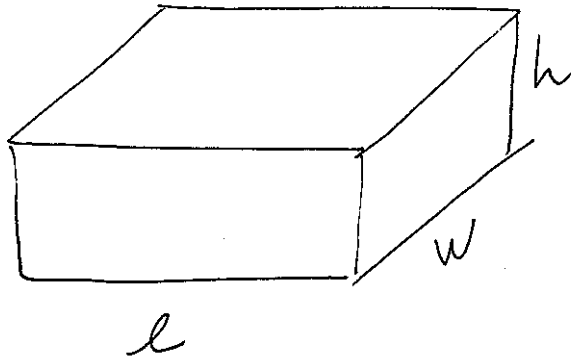
$$6 = x - (-x) + 2(2x) = 6x \Rightarrow x = 1.$$

So the unique critical point is $(1,-1,2)$ just as in the last lecture.

Key Features: ① Algebra easier than before.

② Don't have to solve for one var, which we won't be able to do for complicated g.

Ex: Find the rectangular box of area 6 and largest volume. [What do you expect the ans to be?]



Maximize: $V = lwh$

Subject to: $A = 2lh + 2lw + 2wh = 6.$

$$\nabla V = (wh, lh, lw) = \lambda \nabla A$$
$$= \lambda 2 (h+w, l+h, l+w)$$

$$\Rightarrow \frac{1}{2\lambda} = \frac{1}{w} + \frac{1}{h} = \frac{1}{h} + \frac{1}{l} = \frac{1}{w} + \frac{1}{l}$$

$$\Rightarrow \frac{1}{l} = \frac{1}{w} = \frac{1}{h} \Rightarrow l = w = h$$

Q: Why does this crit pt have to be a max?

Combine with $A = 6$ gives $6l^2 = 6 \Rightarrow l = w = h = 1.$

Point: V and A is very symmetric. In particular if (l_0, w_0, h_0) is a crit pt, so is (w_0, h_0, l_0) and (w_0, l_0, h_0), ... Hence if there is only one crit pt, it must be symmetric.

Why the crit pt $(1,1,1)$ is the abs. max:

Suppose $l \geq 100$ with $A=6$. Then

$$3 = A/2 = lh + lw + hw \geq lh \Rightarrow h \leq 3/l$$

and similarly $w \leq 3/l$. Thus $V = lwh \leq 9/l < 1/10$.

So outside $D = \{A=6 \text{ and } 0 \leq l,w,h \leq 100\}$ when $0 \leq l,w,h$, we have $V < 1/10$. The EVT applies to D and on ∂D have $V < 1/10$ [since on ∂D either one l,w,h is $0 \Rightarrow V=0$ or one l,w,h is $100 \Rightarrow V < 1/10$]

So the abs. max of V on D is 1, occurring at $(1,1,1)$, and this is also the abs max on $\left\{ \begin{matrix} A=6 \\ \text{and} \\ 0 \leq l,w,h \end{matrix} \right\}$

Terms and conditions: To apply Lagrange multipliers, one needs that f and g are differentiable and that $\nabla g \neq \vec{0}$ at every point of the level set $g=c$.

Ex: Minimize $f=y$ subject to $g = y^3 - x^2 = 0$.

Here, abs min of f is 0 at $(0,0)$ but there are no sol'ns to $\{\nabla f = \lambda \nabla g,\ g=0\}$