# Lecture 13: More on min/max (§14.7)

**Last time:** $f: \mathbb{R}^2 \to \mathbb{R}$. Critical point: $\nabla f(a,b) = \vec{0}$

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ and
- $f_{xx}(a,b) > 0 \Rightarrow$ local min
- $f_{xx}(a,b) > 0 \Rightarrow$ local max

If $D < 0 \Rightarrow$ saddle

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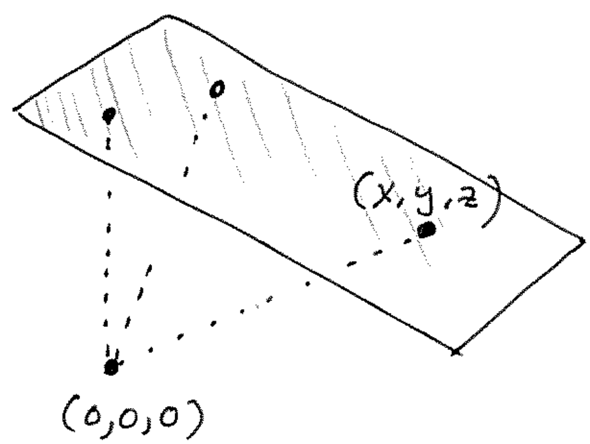
**Ex:** Find the distance from $(0,0,0)$ to the plane

$$x - y + 2z = 6.$$

$$z = \frac{6 - x + y}{2}$$




**Need to minimize:**

$$f(x,y) = \left( \begin{array}{c} \text{distance from} \\ \left(x, y, \frac{6-x+y}{2}\right) \\ \text{to } (0,0,0) \end{array} \right)^2 = x^2 + y^2 + \frac{1}{4}(6 - x + y)^2$$

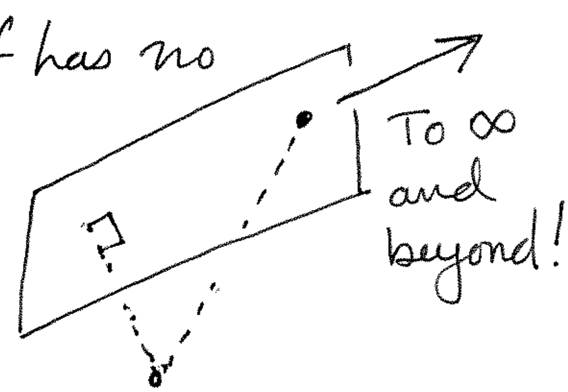
**Critical Points:** $\nabla f = \vec{0}$

$$\frac{\partial f}{\partial x} = 2x + \frac{1}{2}(6 - x + y)\cdot(-1) = \frac{5}{2}x - \frac{1}{2}y - 3$$

$$\frac{\partial f}{\partial y} = 2y + \frac{1}{2}(6 - x + y) = -\frac{1}{2}x + \frac{5}{2}y + 3$$

Setting $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$, we find a unique solution $x = 1$ and $y = -1$. Since this is the only critical point, the minimum distance must be $\sqrt{f(1,-1)} = \sqrt{1^2 + 1^2 + \frac{1}{4}4^2} = \sqrt{6}$.

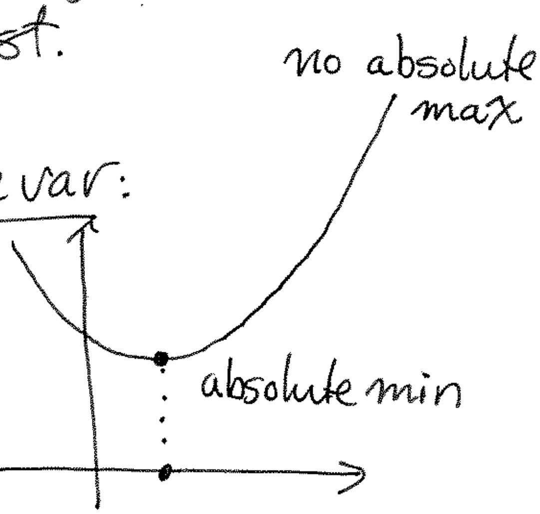
Hey, the same reasoning says that $f$ takes its maximum at $(1,-1)$?! If fact, $f$ has no maximum as is clear geometrically and also algebraically:

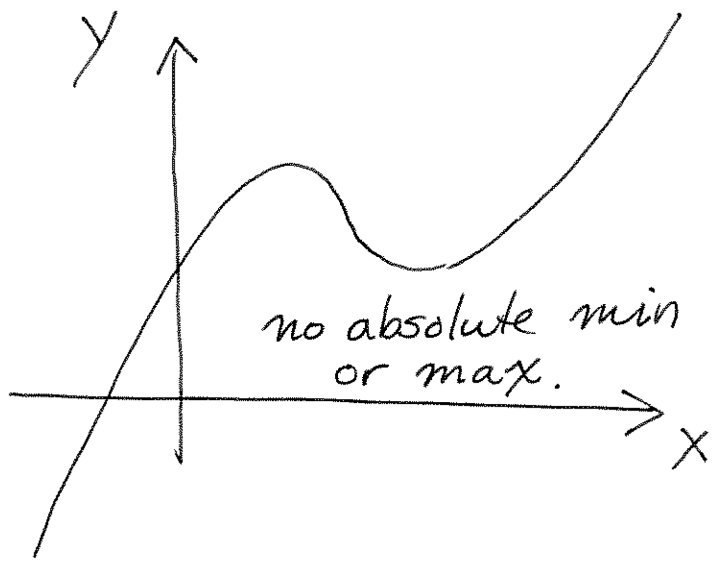



$$f(x,0) = x^2 + \frac{1}{4}(-x)^2 = \frac{5}{4}x^2$$

For this problem, we know there should be a min for geometric reasons, but let's look at some general criteria for absolute min/max to exist.

One var:



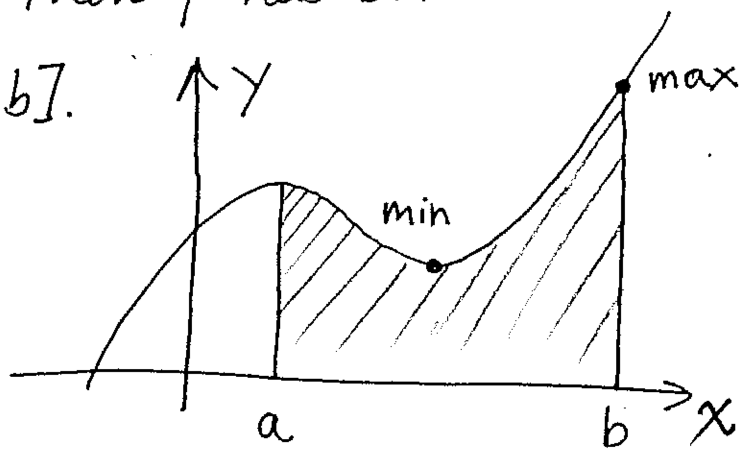



**Extreme Value Theorem:** Suppose $f$ is continuous on $[a,b] = \{a \le x \le b\}$. Then $f$ has both an absolute min and max on $[a,b]$.

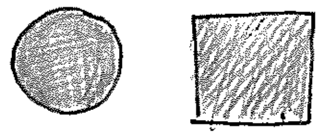
These occure at either

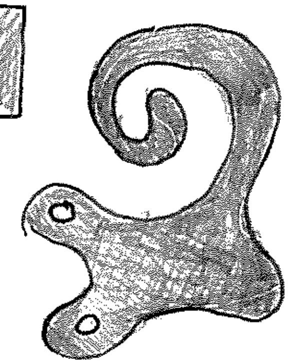
a) a critical point of $f$

b) one of $a$ or $b$.

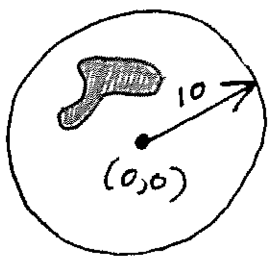



**Two Var:** $D$ subset of $\mathbb{R}^2$:





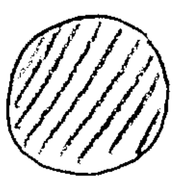
**Bounded:** Contained within a disk:



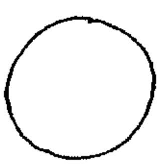

**Closed:** Contains all its boundary points.

**Closed:**

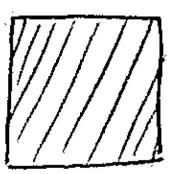
$x^2+y^2 \le 1$



$x^2+y^2$



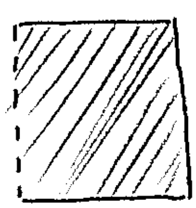
$\begin{Bmatrix} 0 \le x \le 1 \\ 0 \le x \le 1 \end{Bmatrix}$



**Not Closed:**

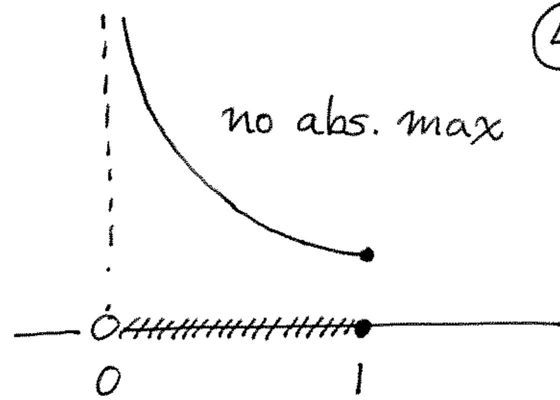
$x^2+y^2 < 1$

$\begin{Bmatrix} 0 < x \le 1 \\ 0 \le y \le 1 \end{Bmatrix}$



Closed important in 1-var too

$f = \frac{1}{x}$ on $(0,1] = \{0 < x \leq 1\}$




Extreme Value Theorem:

Suppose $f$ is continuous on $D$ in $\mathbb{R}^n$. If $D$ is closed and bounded, then $f$ has absolute min and max on $D$. These occure either at critical points of $f$ or on the boundary of $D$.

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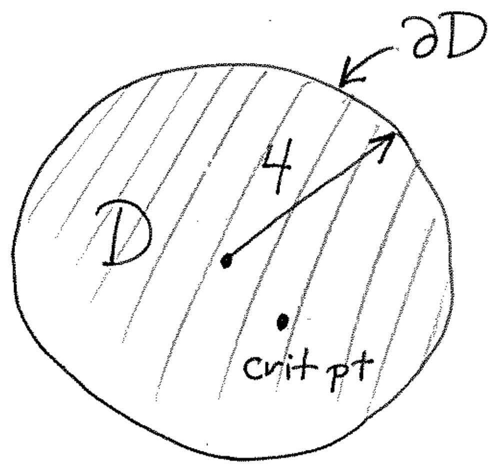
Back to original problem:

$$f(x,y) = x^2 + y^2 + \frac{1}{4}(6 - x + y)^2 \text{ on } \mathbb{R}^2$$

which has one critical point at $(1,-1)$.

Consider $D = \{x^2 + y^2 \leq 16\}$




On $D$, $f$ has one crit pt and there $f = 6$.

On $\partial D$, $f(x,y) \geq x^2 + y^2 = 16$. So 6 is the absolute min value of $f$ on $D$.

Note: $f(x,y) \geq x^2 + y^2 \geq 16$ outside D,
so 6 is the absolute min of $f$ on all of $\mathbb{R}^2$.

Asides: 1) Turns out max on D is $33 + 12\sqrt{2} \approx 50$ and occurs at $(-2\sqrt{2}, 2\sqrt{2})$; will learn how to figure this out next time.

2) Good Check: $f_{xx} = 5/2$, $f_{xy} = -1/2$, $f_{yy} = 5/2$

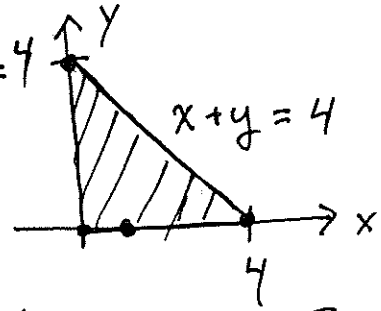
$$D = \begin{vmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{vmatrix} = \frac{25}{4} - \frac{1}{4} = 6 > 0 \text{ and } f_{xx} > 0.$$

So the 2nd derivative test says there is a local min at $(1,-1)$.

3) There are weird functions like $x^2 + y^2(1-x)^3$ which have only one critical point, a local min, but no absolute min!

⑥

Ex: $f = y - xy$  D =




$\nabla f = (-y, 1-x)$

So a unique crit. pt at (1,0). By E.V.T, $f$ has absolute min/max on D, both of which must occur on $\partial D$ as no crit. pts inside D.

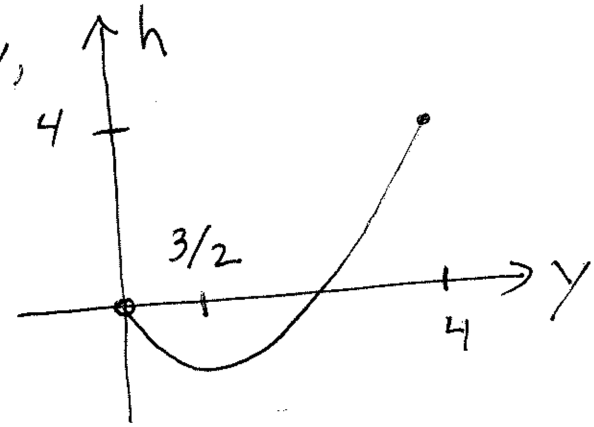
a) Along the bottom: $f(x,0) = 0$ for any $x$.

b) Along the right side: $f(0,y) = y$, so min on this side is 0, max is 4.

c) Along the sloped side we have

$$f(4-y, y) = y - (4-y)y = y^2 - 3y = h(y)$$

Note $h$ has one crit pt, namely, $y = 3/2$, since $h'(y) = 2y - 3$ where $h = -\frac{9}{4} = -2\frac{1}{4}$.




So the absolute max of $f$ on D is 4, occuring at (0,4) and the absolute min is $-9/4$ occuring at $(5/2, 3/2)$.