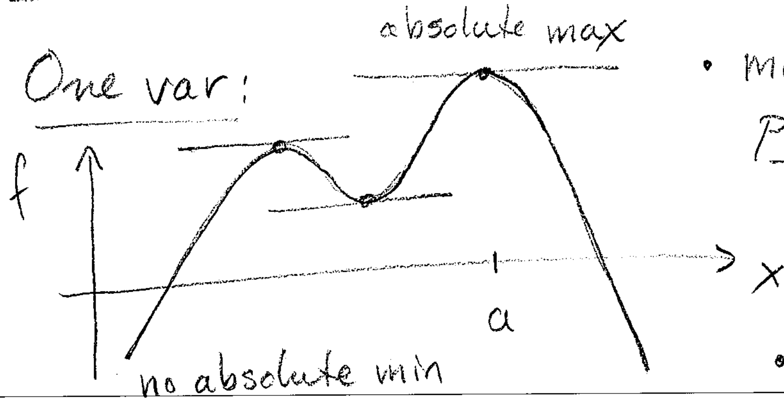
Lecture 12: Intro to min/max (§14.7)

One var:



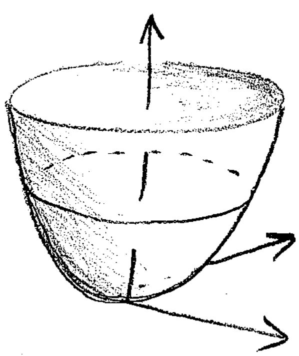
- min/max occur at critical points where $f'(a) \neq 0$
- local vs. absolute extrema.
- 2^{nd} derivative test! $\begin{cases} f''(a) < 0 \Rightarrow \max \\ f''(a) > 0 \Rightarrow \min \end{cases}$

For $f: \mathbb{R}^2 \to \mathbb{R}$, (a,b) is a critical point when

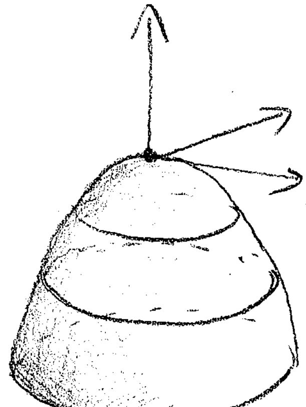
$$\underbrace{\nabla f(a,b) = \big(f_x(a,b),\, f_y(a,b)\big) \text{ is } \vec{0}}_{\text{same as tangent plane is horizontal.}} \text{ or does not exist.}$$

Tale of 3 critial points at $(0,0)$:

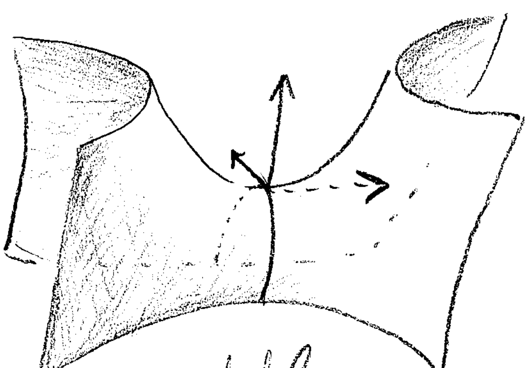
$f(x,y) = x^2 + y^2$	$f(x,y) = -x^2 - y^2$	$f(x,y) = x^2 - y^2$
$\nabla f = (2x, 2y)$	$\nabla f = (-2x, -2y)$	$\nabla f = (2x, -2y)$



local max



local max

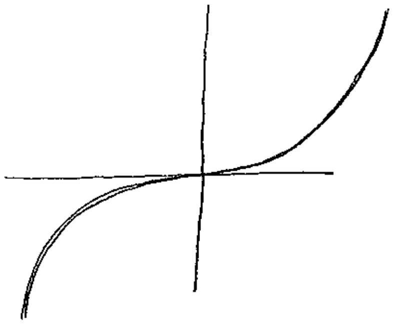


saddle

[Today will give 2^{nd} der. test for such f.

Discuss relationship to ∇f as pointing in the direction of fastest increase in f.]

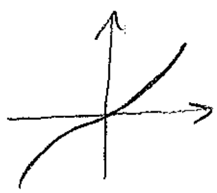
In one var, also a neither case, e.g. $f(x) = x^3$



$$\left.\begin{aligned} f'(x) &= 3x^2 \\ f''(x) &= 6x \end{aligned}\right\} \text{both } 0 \text{ at } 0.$$

This was pretty rare, because its not "stable":

E.g. $f(x) = x^3 + \frac{1}{10}x$ and $x^3 - \frac{1}{10}x$ don't have this issue.



However, saddles *are* stable.

One pt of view on the 2^{nd} der. test in one var.

Taylor series: Near x_0, usually have

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)$$

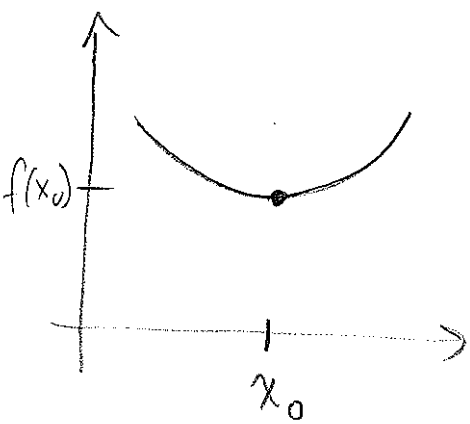
where $E(h)$ is really small, i.e. $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$.

If x_0 is a critical pt, then

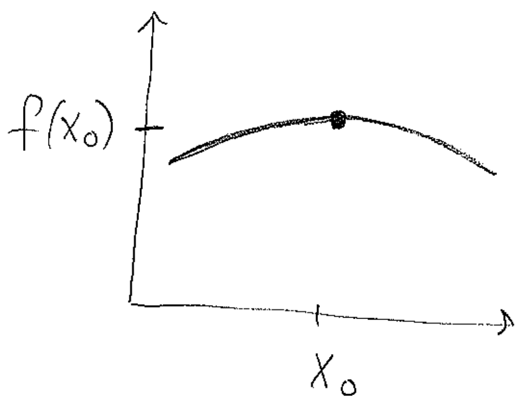
$$f(x_0+h) = f(x_0) + \frac{f''(x_0)}{2}h^2 + E(h)$$

So near x_0 the graph of f looks like

$f''(x_0) > 0$ $\qquad\qquad$ $f''(x_0) < 0$



local min



local max.

<u>Taylor series for $f(x,y)$</u>: For nice functions, have

$$f(x_0+h, y_0+k) = \overbrace{f(x_0,y_0) + f_x(x_0,y_0)h + f_y(x_0,y_0)k}^{\text{linear approx}}$$
$$+ ah^2 + bhk + ck^2 + \underbrace{E(h,k)}_{\text{smaller than other terms.}}$$

next level of accuracy.

Q: What are a, b, c?

A: $a = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $b = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, $c = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0, y_0)$

Reason: Take $x_0 = y_0 = 0$, $x = h, y = k$

$$f(x,y) \approx \underbrace{f(0,0) + f_x(0,0)x + f_y(0,0)y + ax^2 + bxy + cy^2}_{g(x,y)}$$

① $f(0,0) = g(0,0)$

② $f_x(0,0) = g_x(0,0)$ since $g_x(x,y) = f_x(0,0) + 2ax + by$

③ Want
$f_{xx}(0,0) = g_{xx}(0,0)$
as well. Since $g_{xx} = 2a$ this gives $a = \frac{1}{2} f_{xx}(0,0)$.

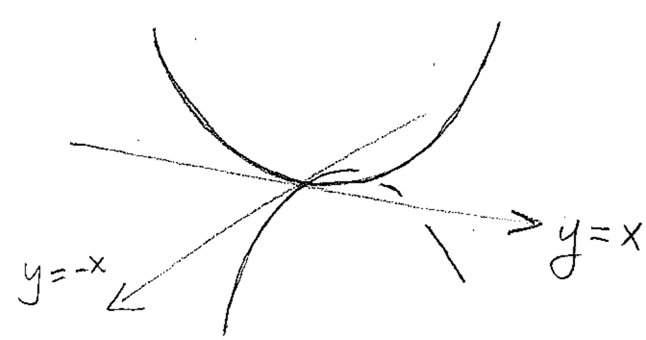
Ex: Why Taylor series are so useful:

$$f(x,y) = \sin\left(\sqrt{1+\frac{x^2}{2+\cos y}} - e^{-xy}\right)$$

Now $(0,0)$ is a crit pt of this mess, but is it a max? Taylor series is:

small compared to the other terms.

$$f(x,y) = \frac{1}{6}x^2 + xy + E(x,y)$$



Look along lines:

$$f(x,x) = \frac{7}{6}x^2 + E(x,x)$$

$$f(x,-x) = \frac{1}{6}x^2 + x(-x) + E(x,-x) = -\frac{5}{6}x^2 + E(x,-x)$$

So f has a saddle at $(0,0)$.

2nd derivative test: Suppose (a,b) is a crit pt of f.

Set

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

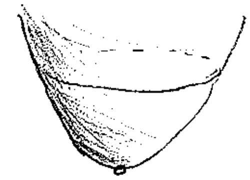
If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local min

If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local max

If $D < 0$ then (a,b) is a saddle.

[If $D = 0$ or $f_{xx}(a,b) = 0$, break glass...]

Ex: $f(x,y) = x^2 + y^2$ $\quad f_x = 2x \quad f_y = 2y$



At $(0,0)$ have

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \text{ and } f_{xx}(0,0) = 2 > 0$$

so a $\boxed{\text{min}}$ ✓

Ex: $f(x,y) = -x^2 - y^2 \quad D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$ and $f_{xx}(0,0) < 0$

so a $\boxed{\text{max}}$ ✓

Ex: $f = x^2 - y^2 \quad D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$

Ex: $\underbrace{f = xy}_{\text{from worksheet.}} \quad D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -4 < 0$

Saddles at $(0,0)$

Under the hood: Changing Coordinates, learn more about on Tuesday.

Higher dims: Eigenvalues etc...