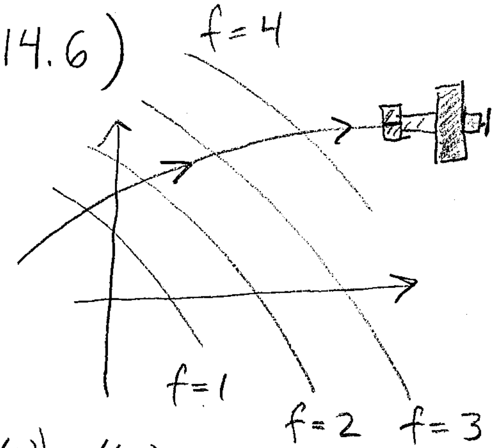
Lecture 11: Directional Derivatives and the Gradient (§14.6)



Last time: Chain Rule

① $f : \mathbb{R}^2 \to \mathbb{R} \quad x, y : \mathbb{R} \to \mathbb{R}$

For $h(t) = f(x(t), y(t))$, we have

$$h'(t) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

② $z = f(x, y)$ with $x = x(t)$ and $y = y(t)$ Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

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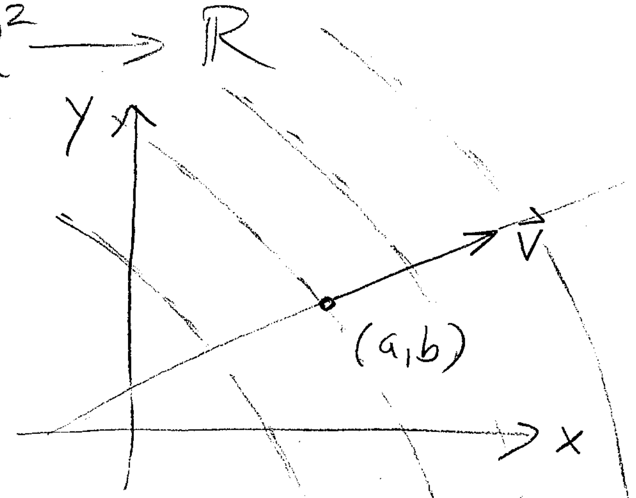
③ Works when $x, y : \mathbb{R}^2 \to \mathbb{R}$ e.g. $x(s,t) = s^2 + t$
$y(s,t) = st + 2$

Now $z(s,t) = f(x(s,t), y(s,t))$

and $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$.

Directional Derivatives: $f : \mathbb{R}^2 \to \mathbb{R}$



$D_{\vec{v}} f(a,b)$ = rate of change in f as we move in direction $\vec{v}$ through (a,b)

$$= \frac{d}{dt} f(\underbrace{a + v_1 t}_{x(t)}, \underbrace{b + v_2 t}_{y(t)}) \Big|_{t=0}$$

where $\vec{v} = (v_1, v_2)$

Ex: $D_{\vec{i}} f(a,b) = \frac{\partial f}{\partial x}(a,b)$

Compute using chain rule (provided f is diff. at (a,b)):

$$D_{\vec{v}} f(a,b) = f_x(x(0), y(0))\, x'(0) + f_y(x(0), y(0)) \cdot y'(0)$$
$$= f_x(a,b)\, v_1 + f_y(a,b)\, v_2.$$

Ex: $f(x,y) = x^2 + y^2$ $\quad \vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

[Here $\vec{u}$ is a unit vector; usually, we take directional derivatives in unit directions since $D_{2\vec{v}} f(a,b) = 2 D_{\vec{v}} f(a,b)$]

$$D_{\vec{u}} f(2,1) = f_x(2,1) \cdot \frac{1}{\sqrt{2}} + f_y(2,1)\left(-\frac{1}{\sqrt{2}}\right)$$
$$= 4 \,{}^1\!/\!\sqrt{2} - 3\,{}^1\!/\!\sqrt{2} = {}^1\!/\!\sqrt{2} \approx 0.7071.$$

Gradient: For $f: \mathbb{R}^2 \to \mathbb{R}$, define

$$\nabla f(a,b) = (f_x(a,b), f_y(a,b))$$

Ex: For $f = x^2 + y^3$, $\nabla f(2,1) = (4,3)$

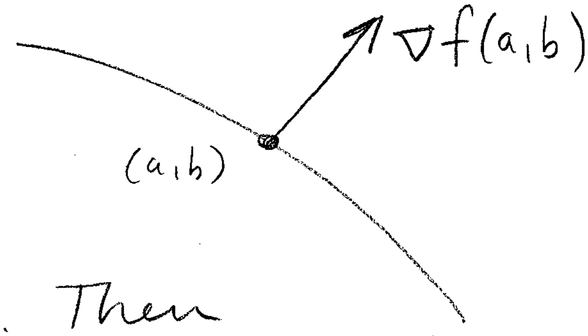
[Will explain geometric meaning shortly, but for now notice

$$D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v}$$

Key props:

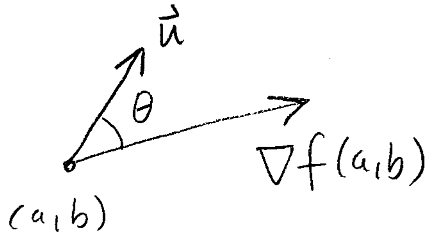
1) $\nabla f(a,b)$ points in the direction that f increases fastest at (a,b). Its length is that rate of increase.

2) $\nabla f(a,b)$ is at right angles to the level set of f at (a,b)



Reasons:

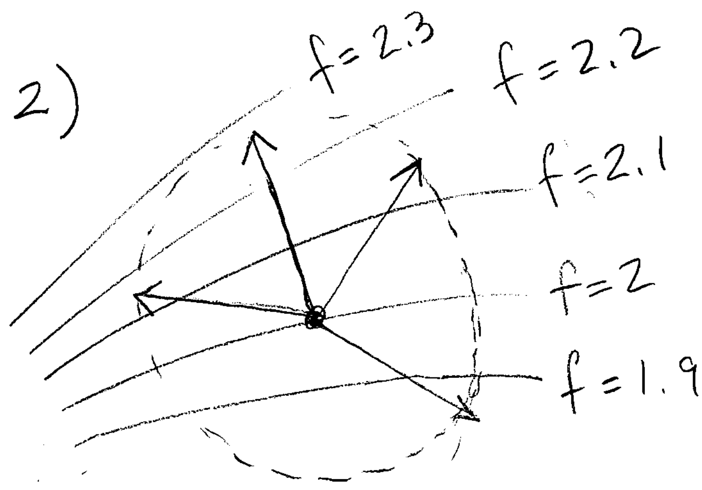
1) Suppose $\vec{u}$ is a unit vector. Then



$$D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$
$$= |\nabla f(a,b)| \, |\vec{u}| \cos\theta$$

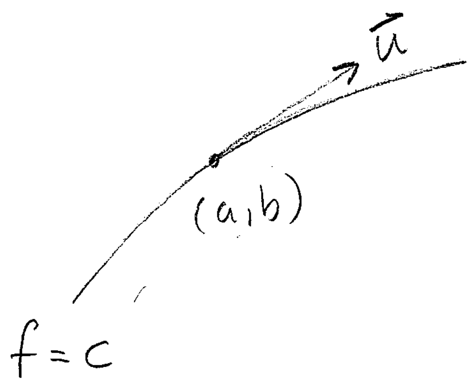
This is largest when $\theta = 0$, i.e. $\vec{u}$ has the same direction as $\nabla f(a,b)$, and, for that $\vec{u}$, have $D_{\vec{u}} f(a,b) = |\nabla f(a,b)|$.

2)



To cross as many level sets as possible with a unit vector, go at right angles to the level sets.

2) Alternate reason: Suppose $\vec{u}$ is tangent to the level set of f at (a,b).



Since f is constant as we move along the level set,

$$0 = D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$$

Thus ∇f and $\vec{u}$ are $\perp$

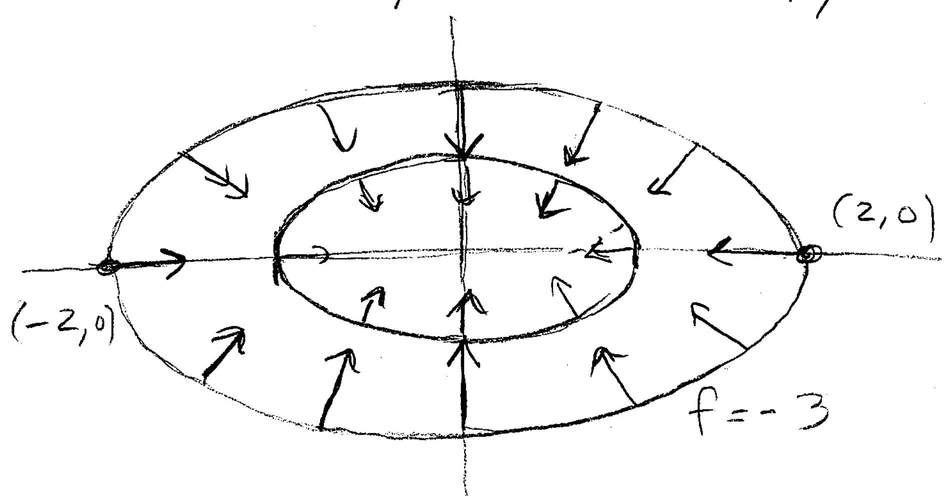
Ex: $f(x,y) = 1 - x^2 - 4y^2 \quad \nabla f = (-2x, -8y)$

[Go to computer demo!]

Level sets: $f=0: 1 - x^2 - 4y^2 = 0 \iff x^2 + 4y^2 = 1$

$f=-3: 1 - x^2 - 4y^2 = -3 \iff x^2 + 4y^2 = 4$

$\iff \dfrac{x^2}{4} + y^2 = 1$



Gradient:

$\nabla f(1,0) = (-2,0)$

$\nabla f(0,1) = (0,-8)$

Also have ∇f for $f : \mathbb{R}^3 \to \mathbb{R}$ where
$\nabla f(P) = (f_x(P), f_y(P), f_z(P))$. Has same properties.

<u>Q</u>: Find the tangent plane to the sphere
$x^2+y^2+z^2 = 6$ at $(1,1,2) = P$.

<u>A</u>: Take $f(x,y,z) = x^2+y^2+z^2$ so sphere is $\{f=6\}$.
Then $\nabla f = (2x, 2y, 2z)$ and is $\perp$ to the level set, we use $\vec{n} = \nabla f(P) = (2,2,4)$ as the normal to the plane. So, our eqn is

$$2(x-1)+2(y-1)+4(z-2) = 0$$

$$\iff x+y+2z = 6.$$