Lecture 10: Chain Rule (§14.5)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if

$$f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\,k + E(h,k)$$

where
$$\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0$$

Thm: If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous near (a,b) the f is differentiable at (a,b).

Ex: $f(x,y) = -x^2 - y^2 \quad f_x = -2x \quad f_y = -2y$

$\Rightarrow f$ is differentrable everywhere.

Alt. notation: $\Delta x = h = x - a$
$\Delta y = k = y - b$
[Read in words ↙]

$$\Delta f = f(x,y) - f(a,b) \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y$$

↑ approximately

You all know:
$$\frac{d}{dt}\sin(t^2) = \cos(t^2)\cdot(2t)$$

[Today, will discuss the analog for several variables.]

Setup: For $f, g : \mathbb{R} \to \mathbb{R}$, consider the composition

$h = f \circ g$, where $h(t) = f(g(t))$.

Ex: $f(t) = \cos(t)$, $g(t) = t^2 \Rightarrow h(t) = \cos(t^2)$

Chain Rule: $h'(t) = f'(g(t))\,g'(t)$.

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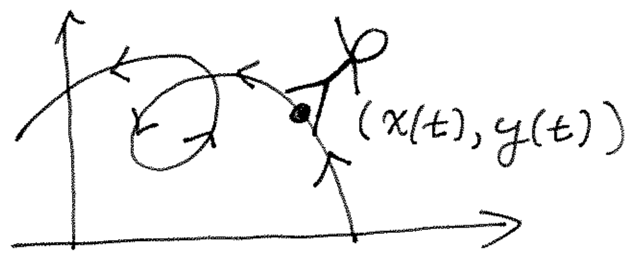
① $f : \mathbb{R}^2 \to \mathbb{R}$



② parameterized curve

$(x(t), y(t))$

③ Compose them: $h(t) = f(x(t), y(t))$ so $h : \mathbb{R} \to \mathbb{R}$

Ex: f = temperature as a function of position.

h = temperature as a function of time.

Q: What is $h'(t)$? [In terms of the derivatives of x, y and the partials of h.]

That is, we want to find the linear approximation

$$h(t + \Delta t) = h(t) + h'(t)\Delta t + \underbrace{E(\Delta t)}_{\text{small compared to } \Delta t.}$$

Know

$$x(t+\Delta t) = x(t) + x'(t)\,\Delta t + E_1(t)$$
$$y(t+\Delta t) = y(t) + y'(t)\,\Delta t + E_2(t)$$

and

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y$$

So plug in and get

$$h(t+\Delta t) = f(x(t+\Delta t), y(t+\Delta t))$$
$$= f(\underbrace{x(t)}_{x} + \underbrace{x'(t)\Delta t + E_1(t)}_{\Delta x}, \underbrace{y(t)}_{y} + \underbrace{y'(t)\Delta t + E_2(t)}_{\Delta y})$$
$$\approx f(x(t), y(t)) + f_x(x(t), y(t))\big(x'(t)\Delta t + E_1(t)\big)$$
$$+ f_y(x(t), y(t))\big(y'(t)\Delta t + E_2(t)\big)$$
$$\approx h(t) + \big(f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t)\big)\Delta t$$

where I've thrown away all the terms with $E_1(t)$ and $E_2(t)$.

__Chain Rule:__ $h(t) = f(x(t), y(t))$. Then

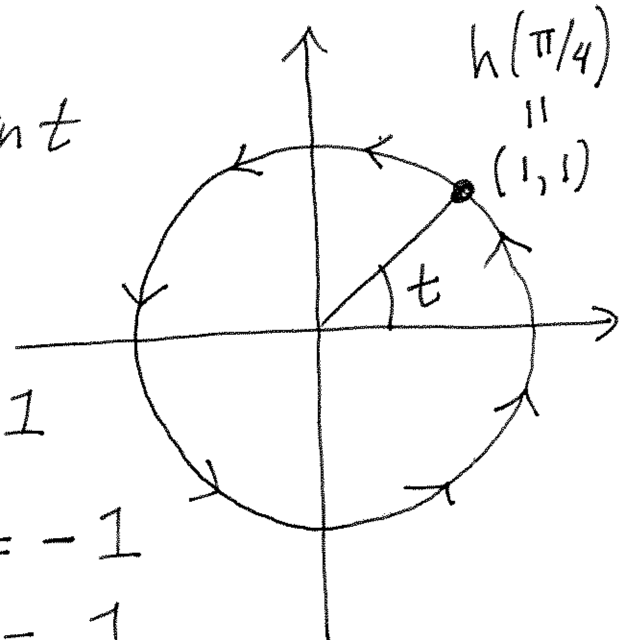
$$h'(t) = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t)$$

[Read in words.]

Ex: $f(x,y) = (x+y^2)^2$

$x(t) = \sqrt{2}\cos t \quad y(t) = \sqrt{2}\sin t$

$h(t) = f(x(t), y(t))$



Find $h'(\pi/4)$: $x(\pi/4) = y(\pi/4) = 1$

$x'(t) = -\sqrt{2}\sin t \quad x'(\pi/4) = -1$

$y'(t) = +\sqrt{2}\cos t \quad y'(\pi/4) = 1$

$f_x = 2(x+y^2) \quad f_y = 2(x+y^2)(2y) = 4y(x+y^2)$

So:

$$h'(\pi/4) = f_x(x(\pi/4), y(\pi/4)) \cdot x'(\pi/4)$$
$$+ f_y(x(\pi/4), y(\pi/4)) \cdot y'(\pi/4)$$
$$= f_x(1,1)(-1) + f_y(1,1)\cdot(1)$$
$$= 4(-1) + 8(1) = 4.$$

Note: $h(t) = (2\cos t + 2\sin^2 t)^2$

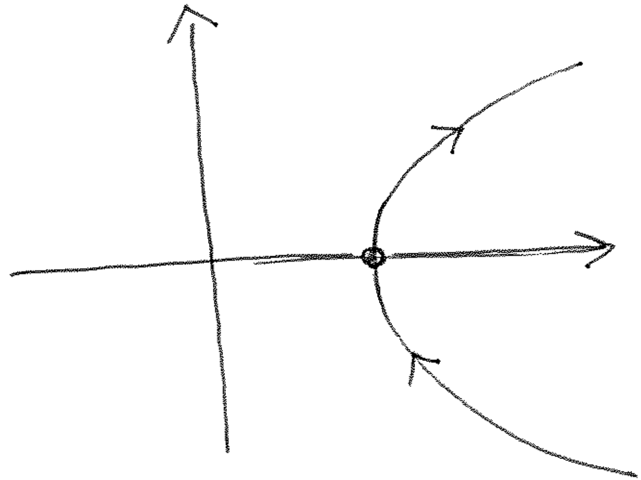
so you can double-check directly in this case...

Ex: $f(x,y) = (x+3y)^2$

$x(t) = t^2+1$

$y(t) = t$

$h(t) = f(x(t), y(t))$.



Find $h'(0)$ using the Chain Rule:

$x'(t) = 2t \qquad x'(0) = 0$

$y'(t) = 1 \qquad y'(0) = 1$

$f_x = 2(x+3y) \qquad f_y = 2(x+3y)^2 \cdot 3 = 6(x+3y)^2$

So:

$$h'(0) = f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0))\, y'(0)$$

$$= f_x(1,0) \cdot 0 + f_y(1,1) \cdot 1$$

$$= 0 + 6 \cdot 1 = 6.$$

Note: $h(t) = ((t^2+1)+3t)^2 = (t^2+3t+1)^2$

So can double check this directly...

Alternate viewpoint:

$f, g: \mathbb{R} \to \mathbb{R}$ and $h(t) = f(g(t))$

Set $y = f(x)$ and $x = g(t)$ so that y can be viewed as a function of t. Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \qquad \text{"Cancel the } dx\text{'s"}$$

Compare

$$\frac{dh}{dt}(t) = \frac{df}{dx}(g(t)) \cdot \frac{dg}{dt}(t)$$
$$= f'(g(t))\, g'(t)$$

Now suppose

$$h(t) = f(x(t), y(t))$$

Chain Rule: $$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ "Cancel ∂x with dx."

Sometimes write $z = f(x, y)$ and $x = x(t)$, $y = y(t)$ so that z becomes a function of t and:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Also works for $f: \mathbb{R}^n \to \mathbb{R}$ for $n > 2$:

<u>Ex</u>: $w = f(x, y, z) = x^2 + yz$

$$x(t) = t$$
$$y(t) = t^2$$
$$z(t) = 1 - t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= 2x \cdot 1 + z\,(2t) + y(-1)$$
$$= 2t + (1-t)(2t) + t^2(-1)$$
$$= 4t - 3t^2$$

<u>Check</u>: $w(t) = t^2 + t^2(1-t) = 2t^2 - t^3$

$w'(t) = 4t - 3t^2.$