Math 241: Honors HW 4 due Wednesday, December 11, 2024.

Overview: Each problem indicates the lecture/discussion after which you will have seen enough to solve the problem.

Rules: While everything you need to do these problems was covered in class or is in our textbook, you may consult any source you find useful as you work on them. Collaboration on homework is permitted, nay encouraged. However, you must write up your solutions individually and understand them completely.

Mechanics: Your solutions should be turned in, on paper, at the beginning of the last class.

Problems:

1. (Nov. 6) Let S be the surface defined by

$$z = \frac{1}{\sqrt{x^2 + y^2}} \quad \text{for } z \geq 1.$$

   (a) Sketch the graph of this surface.

   (b) Use an improper integral to show that the volume of the region E bounded by S and the plane $z = 1$ is finite.

   (c) Use an improper integral to show that the area of S is infinite.

2. (Nov. 8) Let a be a positive constant. Use Green's Theorem to calculate the area under one arch of the cycloid given by $x = a(t - \sin t)$ and $y = a(1 - \cos t)$.

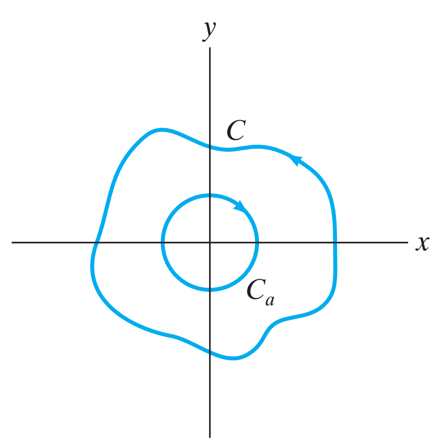
3. (Nov. 8) Consider the vector field $\mathbf{F} = \frac{1}{x^2+y^2} \langle -y,\, x \rangle$.

   (a) Draw a sketch of this vector field. Do the arrows get shorter or longer as you approach the origin?

   (b) Fix $0 < a < 1$. Verify Green's theorem for the annular region $A = \{a^2 \leq x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$ by calculating all the relevant integrals.

   (c) Now let D be the unit disk in $\mathbb{R}^2$. Check directly whether or not the formula for Green's theorem holds for D. Explain why your answer makes sense theoretically.

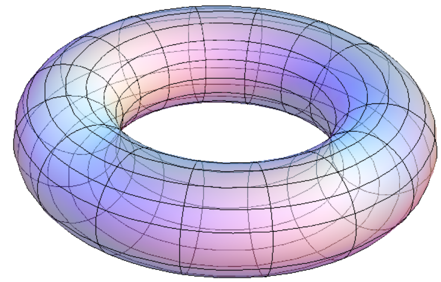
   (d) Suppose C is any simple closed curve in $\mathbb{R}^2$ lying outside the circle $C_a = \{x^2 + y^2 = a^2\}$ for some $a > 0$. Suppose C bounds a region that contains the origin, as shown for example at right. Argue that if C is oriented counterclockwise then $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$.



   (e) Suppose C is as above but that the region bounded by C does not contain the origin. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in this case.

   (f) Determine whether $\mathbf{F}$ is conservative.

4. (Nov. 18) Let T be the surface obtained by rotating the circle $(x-3)^2+z^2=1$ in the xz-plane about the z-axis. This surface is called a torus and shown at right. The inner radius is 2 and the outer radius is 4, making the radius of the tube itself equal to 1.



   (a) Find a parameterization $\mathbf{r}$ for T, being sure to specify it domain. Explain what the geometric parameters are and why the formula you give works.

   (b) Compute the volume of T by computing the flux of some vector field $\mathbf{F}$.

   (c) Compute the volume of T by a 3-dimensional change of coordinates where your final integral is over a rectangular box.

5. (Nov. 20) Suppose D is region in $\mathbb{R}^3$.

   (a) Suppose $f\colon D \to \mathbb{R}$ is a function with continuous second order partial derivatives. Show that the vector field $\operatorname{curl}(\nabla f)$ is $\mathbf{0}$ on D.

   (b) Suppose $\mathbf{F}$ is a vector field on the region D whose component functions have continuous second order partial derivatives. Show that the function $\operatorname{div}(\operatorname{curl}\mathbf{F})$ is 0 on D.

6. (Nov. 20) Suppose $\mathbf{F}$ is a vector field on a region D in $\mathbb{R}^3$ whose component functions have continuous second order partial derivatives.

   (a) Suppose D is a closed bounded region in $\mathbb{R}^3$. Show that the flux of $\operatorname{curl}\mathbf{F}$ through ∂D is 0 using the Divergence Theorem.

   (b) Suppose S is any connected closed orientable[1] surface S contained in D. Show that the flux of $\operatorname{curl}\mathbf{F}$ through S is 0 using Stokes' theorem. Hint: Divide S into surfaces A and B meeting along a circle C. You can take A to be a small disc on S.

7. (Dec. 2) Throughout, $\mathbf{F}$ is a vector field defined on a region D in $\mathbb{R}^3$ with continuous second order partial derivatives.

   (a) Suppose D is $\mathbb{R}^3$ itself. Then we learned in class that $\mathbf{F}$ is conservative if and only if $\operatorname{curl}\mathbf{F} = \mathbf{0}$. Similarly, it turns out that there exists a vector field $\mathbf{G}$ with $\operatorname{curl}\mathbf{G} = \mathbf{F}$ if and only if $\operatorname{div}\mathbf{F} = 0$. (Compare Problem 5 above.) For $\mathbf{F} = \langle z, x, e^y \rangle$, find $\mathbf{G}$ with $\operatorname{curl}\mathbf{G} = \mathbf{F}$.

   (b) From class, we know

$$\mathbf{F} = \frac{1}{|\mathbf{r}|^3}\mathbf{r} = \frac{1}{(x^2+y^2+z^2)^{2/3}}\langle x, y, z \rangle$$

   on $D = \{(x,y,z) \neq \mathbf{0}\}$ has $\operatorname{div} F = 0$ on D. Use Problem 6(b) to argue that $\mathbf{F}$ is not $\operatorname{curl}\mathbf{G}$ for any $\mathbf{G}$ on D.

[1] It's a topological fact that every closed surface in $\mathbb{R}^3$ is orientable.