Math 241: Honors HW 3 due Thursday, November 7, 2024.

Overview: Each problem indicates the lecture/discussion after which you will have seen enough to solve the problem.

Rules: While everything you need to do these problems was covered in class or is in our textbook, you may consult any source you find useful as you work on them. Collaboration on homework is permitted, nay encouraged. However, you must write up your solutions individually and understand them completely.

Mechanics: Your solutions should be turned in, on paper, at the beginning of your discussion section. Your answers do not need to be handwritten — typed answers are very welcome — but you do need to turn in a physical copy on the due date.

Problems:

1. (Oct. 4) Let C be the curve in $\mathbb{R}^3$ given by the parameterization $\mathbf{r}(t) = \langle e^{-t}\cos t,\ e^{-t}\sin t,\ e^{-t} \rangle$ for $t \geq 0$.

   (a) Describe and sketch the curve C. Hint: Find a surface on which it lies.

   (b) Calculate the arc length function $s(t)$, which gives the length of the segment of C between $\mathbf{r}(0)$ and $\mathbf{r}(t)$ as a function of t for all $t \geq 0$.

   (c) Express the original t in terms of s and use this to define a new parameterization $\mathbf{f}\colon [0, L] \to \mathbb{R}^3$ of C where $|\mathbf{f}'(s)| = 1$ for all s, where L is the total length of C.

   (d) Make sure your $\mathbf{f}$ is defined at L and explain why it is continuous there.

2. (Oct. 4) Suppose C is a smooth curve in $\mathbb{R}^3$ that does not pass through a point P. Assume that A is the point on C that is closest to P. Show that the unit tangent vector $\mathbf{T}$ to C at P is orthogonal (perpendicular) to $\overrightarrow{PA}$.

3. (Oct. 14) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are differentiable functions of one variable. Consider the vector field $\mathbf{F} = \langle f(x) + y,\ x + g(y) \rangle$, which is defined on all of $\mathbb{R}^2$.

   (a) Show that $\mathbf{F}$ is conservative without producing a potential function.

   (b) Find a potential function $h\colon \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla h = \mathbf{F}$. Your answer will involve integrals of f and g; these should be **definite integrals only** so that e.g. $h(1, -2)$ is an actual number.

   (c) Give a formula for $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is any path from (x_0, y_0) to (x_1, y_1).

4. (Oct. 25) Let R be the intersection of the solid cylinders $x^2 + z^2 \leq 1$ and $y^2 + z^2 \leq 1$.

   (a) Draw a picture of R. Hint: The result is symmetric with respect to rotation about the z-axis by $\pi/2$.

   (b) Compute the volume of R.

5. (Oct. 30) A fundamental tool for constructing functions, especially parameterizations and transformations, is the notion of a convex combination. The basic example is our parameterization of the line segment from A to B in $\mathbb{R}^n$ by:

$$\mathbf{r}(t) = (1-t)A + tB \quad \text{for } 0 \le t \le 1.$$

Here $\mathbf{r}(0) = A$ and $\mathbf{r}(1) = B$ and e.g. $\mathbf{r}(1/2)$ is the point halfway between A and B.

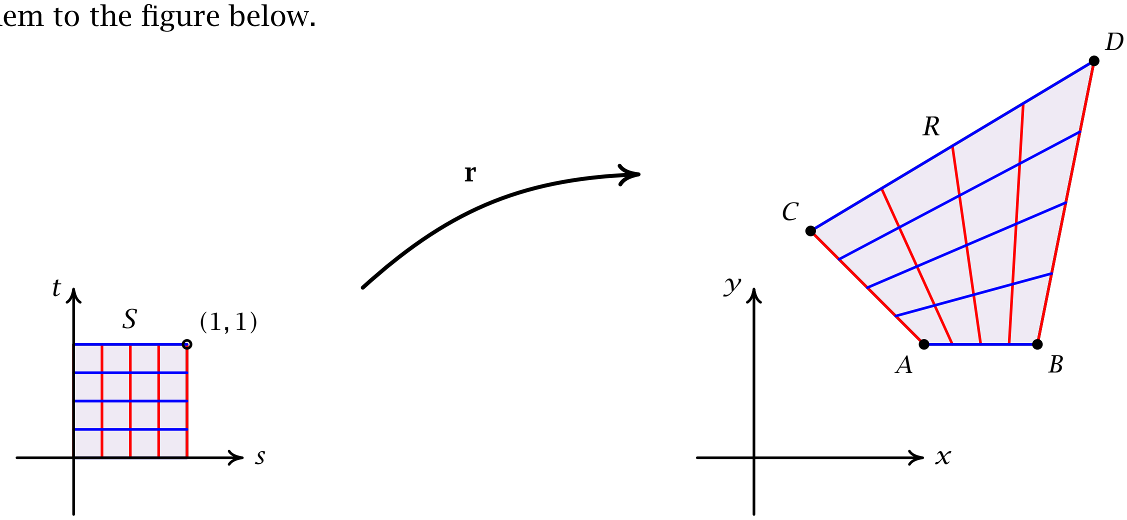
Given points A, B, C, and D in $\mathbb{R}^n$, consider the functions

$$\mathbf{r}(s,t) = (1-t)\big((1-s)A + sB\big) + t\big((1-s)C + sD\big) \tag{1}$$
$$\mathbf{f}(s,t) = (1-s)\big((1-t)A + tC\big) + s\big((1-t)B + tD\big) \tag{2}$$

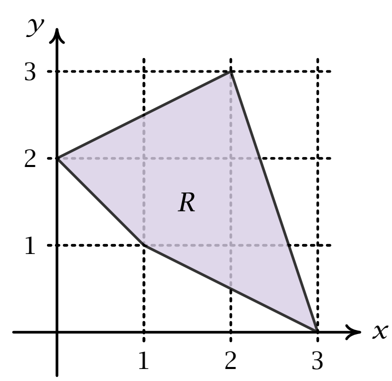
for $0 \le s, t \le 1$.

(a) Check that $\mathbf{r}$ and $\mathbf{f}$ are the same function.

(b) In the case of $n = 2$, explain what the formulas for $\mathbf{r}$ and $\mathbf{f}$ mean geometrically and relate them to the figure below.

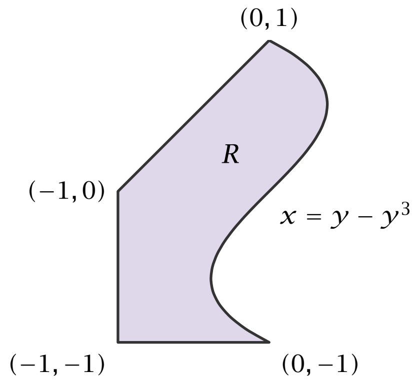


(c) Consider the region R shown below right. Compute the area of R and find its centroid by evaluating the needed integrals over the unit square S by using a change of coordinates modeled on the function $\mathbf{r}$.



(d) Check your answer in (c) by breaking R up into simple (Type I and Type II) regions.

6. (Oct. 16) Consider the region R shown below.



(a) Use ideas from Problem 5 to find a change of coordinates transformation $T\colon S \to R$ where $S = \{0 \le s, t \le 1\}$ is the unit square.

(b) Use your answer in (a) to compute the area of R. Be sure to check your answer against more direct computation.

7. (Oct. 23) The function $f(x, y) = 1/\sqrt{xy}$ is unbounded when x or y is 0. So on the unit square $D = \{0 \le x, y \le 1\}$, the integral $\iint_D f\,dA$ is an *improper double integral*, analogous to $\int_0^1 1/\sqrt{x}\,dx$.

(a) For $0 < \epsilon < 1$ and $0 < \delta < 1$, consider the subsquare $D_{\epsilon,\delta} = \{\epsilon \le x \le 1,\ \delta \le y \le 1\}$ of D and define:

$$I(\epsilon, \delta) = \iint_{D_{\epsilon,\delta}} \frac{1}{\sqrt{xy}}\,dA$$

Evaluate the limit $\lim_{(\epsilon,\delta)\to(0,0)} I(\epsilon, \delta)$, which is a finite value. We then say this improper integral converges and define the limit as the value of $\iint_D f\,dA$.

(b) In the same manner, determine if each of the improper integrals

$$\iint_D \frac{1}{x+y}\,dA \quad \text{and} \quad \iint_D \frac{x}{y}\,dA$$

converges, and, if so, find its value.