Math 241: Honors HW 2 due Thursday, October 10, 2024.

Overview: Each problem indicates the lecture/discussion after which you will have seen enough to solve do the problem.

Rules: While everything you need to do these problems was covered in class or is in our textbook, you may consult any source you find useful as you work on them. Collaboration on homework is permitted, nay encouraged. However, you must write up your solutions individually and understand them completely.

Mechanics: Your solutions should be turned in, on paper, at the beginning of your discussion section. Your answers do not need to be handwritten — typed answers are very welcome — but you do need to turn in a physical copy on the due date.

Course webpage: http://dunfield.info/241

Problems:

1. (Sept. 18) Suppose d is a non-negative integer. A function $f\colon \mathbb{R}^3 \to \mathbb{R}$ is said to be *homogenous of degree d* when for all (x, y, z) in $\mathbb{R}^3$ and t in $\mathbb{R}$ one has $f(tx, ty, tz) = t^d f(x, y, z)$.

   (a) Determine which of the following functions are homogenous. For those that are, indicate the degree.
   $$f = x^3 + xy^2 - 6y^3 \qquad g = x^3y - x^2z^2 + z^8 \qquad h = 2$$

   (b) Suppose $f\colon \mathbb{R}^3 \to \mathbb{R}$ is homogenous of degree d. Assuming that f is differentiable everywhere, use the Chain Rule to show that it satisfies:
   $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = d \cdot f$$

2. (Sept. 19) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by
   $$f(x,y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

   (a) For $(x, y) \neq 0$, a straightforward calculation shows $f_x = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$. Also for $(x, y) \neq 0$, find the formula for f_y, which is quite similar.

   (b) Compute $f_x(0,0)$ and $f_y(0,0)$ directly from limit definition of the partial derivative.

   (c) Show that f_x and f_y are both continuous on all of $\mathbb{R}^2$. What does that tell us about where f is differentiable?

   (d) Calculate $f_{xy}(0,0)$ and $f_{yx}(0,0)$ and show they are different. (Hint: Both are integers.) What must be going on so that Clairaut's Theorem in Section 14.3 is not violated?

   Remark: The function f has the symmetry $f(y, x) = -f(x, y)$ which explains how $f_{xy}(0,0)$ and $f_{yx}(0,0)$ differ in (d).

3. (Sept. 20)

   (a) Suppose S in $\mathbb{R}^3$ is the sphere given by $x^2 + y^2 + z^2 = r^2$. For any $P = (a, b, c)$ on S, show that a normal to the tangent plane to S at P is $\mathbf{v} = \overrightarrow{OP}$ where $O = (0, 0, 0)$ is the origin.

   (b) Consider the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = (x^2 + y^2 + z^2 - 1)^2$. The level set $L = \{f = 0\}$ is the unit sphere. What happens when we try to use ∇f to find the tangent plane to P on L?

4. (Sept. 25) Consider $f(x, y) = e^{xy+2x}$.

   (a) Find the linear approximation of f at $(0, 0)$ and use it to estimate $f(0.1, 0.3)$.

   (b) Find the second-order Taylor polynomial of f at $(0, 0)$ and use it to give a better estimate for $f(0.1, 0.3)$.

   (c) Compare your estimates in (a) and (b) with the numerical approximation to $f(0.1, 0.3)$ given by your calculator.

5. (Sept. 27)

   (a) Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. Argue that if f has at least two local maxima then it must also have a local minimum.

   Hint: Let $a < b$ be critical points corresponding to two local maxima, and apply the Extreme Value Theorem to $[a, b]$.

   (b) In contrast, consider the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by

   $$g(x, y) = 2 - (xy^2 - y - 1)^2 - (y^2 - 1)^2.$$

   Show that this has exactly two critical points, and both are local maxima.

6. (Sept. 30) Fix a length $p > 0$. Let T in $\mathbb{R}^3$ consist of those (x, y, z) corresponding to the sides of a triangle with perimeter p. Concretely, this is equivalent to satisfying the equations:

   $$x \geq 0 \qquad y \geq 0 \qquad z \geq 0 \qquad x + y + z = p \qquad z \leq x + y \qquad x \leq y + z \qquad y \leq x + z$$

   (a) Show that T itself is an equilateral triangle. What kind of triangles correspond to the edges of T? How about the vertices?

   (b) Heron's formula for the area of a triangle whose sides have lengths x, y, and z is

   $$A = \sqrt{s(s - x)(s - y)(s - z)} \quad \text{where } s = p/2.$$

   Use Lagrange multipliers to find the maximum of A on T and so show that the triangle of largest area with a fixed perimeter p is equilateral. Remember to consider the possibility that the maximum of A is on the boundary of T.