Math 241: Honors HW 1 due Thursday, September 19, 2024.

Overview: The problems here are more extended and in-depth than the ones in the regular online HW, or have answers that cannot easily be entered into WebAssign. Each problem indicates the lecture/discussion after which you will have seen enough to solve do the problem.

Rules: While everything you need to do these problems was covered in class or is in our textbook, you may consult any source you find useful as you work on them. Collaboration on homework is permitted, nay encouraged. However, you must write up your solutions individually and understand them completely.

Mechanics: Your solutions should be turned in, on paper, at the beginning of your discussion section. Your answers do not need to be handwritten — typed answers are very welcome — but you do need to turn in a physical copy on the due date.

Course webpage: http://dunfield.info/241

Problems:

1. (Aug. 30) Suppose P is a parallelogram in $\mathbb{R}^2$ where all four sides have equal length. Use vector methods to verify that the diagonals of P are perpendicular.

2. (Aug. 30) Use the formula $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta$ for vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ to derive:

   (a) The Cauchy-Schwartz Inequality $\mathbf{v} \cdot \mathbf{w} \leq |\mathbf{v}||\mathbf{w}|$

   (b) The Triangle Inequality $|\mathbf{v} + \mathbf{w}| \leq |\mathbf{v}| + |\mathbf{w}|$. *Hint:* use $|\mathbf{v} + \mathbf{w}|^2 = (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w})$.

   (c) Give a geometric interpretation of the Triangle Inequality that explains the name.

3. (Sept. 4) Suppose $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in $\mathbb{R}^3$. Assuming the cross product satisfies Theorems 8, 9, 10, and 11 of Section 12.4 of Stewart as well as the right-hand rule, this tells us the cross products of all the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Expand

   $$\mathbf{v} \times \mathbf{w} = (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) \times (w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k})$$

   using these properties and see they force the formula used in class to define the cross product:

   $$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \mathbf{k}$$

   This helps explain the pesky minus sign on the $\mathbf{j}$ term: it comes from the underlying geometric properties of the cross product.

4. (Sept. 4) Let P be a point in $\mathbb{R}^3$.

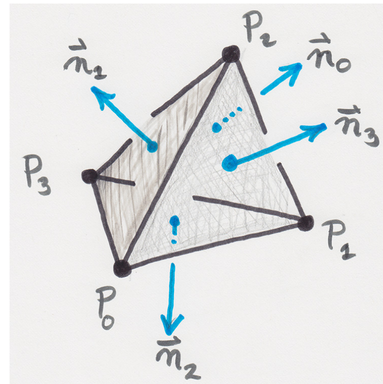
   (a) Suppose P is not on the line L which passes through points Q and R. Argue that the distance d from P to the line L is

   $$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} \quad \text{where } \mathbf{a} = \overrightarrow{QR} \text{ and } \mathbf{b} = \overrightarrow{QP}.$$

(b) Suppose P is not on the plane W passing through points Q, R, and S. Deduce that the distance from P to the plane W is:

$$d = \frac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} \quad \text{where } \mathbf{a} = \overrightarrow{QR},\ \mathbf{b} = \overrightarrow{QS}, \text{ and } \mathbf{c} = \overrightarrow{QP}.$$

5. (Sept. 4) Suppose P_0, P_1, P_2, P_3 are four points in $\mathbb{R}^3$ that do not lie in the same plane. Join each pair of points by a straight line segment and fill in the resulting triangles to make the boundary of a solid called a tetrahedron; see the picture at right. For the triangular face F_i opposite to the vertex P_i, let $\mathbf{n}_i$ be a vector that is normal to F_i, points out from the tetrahedron, and whose length is equal to the area of F_i. Show that $\mathbf{n}_0 + \mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3 = 0$.



6. (Sept. 6) For each of the functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ below, determine several level curves of the given function f and draw these as a single contour plot (make sure to indicate the height c of each curve). Use the contour plot together with some vertical slices (e.g. what happens over the x-axis) to make a sketch of the graph of f.

(a) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$.

(b) $f(x, y) = \sin(x + y)$.

7. (Sept. 12) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x, y) = \frac{\sin(x)y^2}{x^2 + y^2} \quad \text{for } (x, y) \neq \mathbf{0}$$

In this problem, you'll show $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$.

(a) For $\epsilon = 1/2$, find some $\delta > 0$ so that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$. Hint: As with the example in class, the key is to relate $|x|$ and $|y|$ with $|\mathbf{h}|$.

(b) Repeat with $\epsilon = 1/10$.

(c) Now show that $\lim_{\mathbf{h}\to\mathbf{0}} f(\mathbf{h}) = 0$. That is, given an arbitrary $\epsilon > 0$, find a $\delta > 0$ so that that when $0 < |\mathbf{h}| < \delta$ we have $|f(\mathbf{h})| < \epsilon$.

(d) Explain why the limit laws that you learned in class aren't enough to compute this particular limit.