Lecture 8: Sylow Theorems. (§4.5 of [DF], §29-34 of [R1])

Last time:

Sylow Thms: G finite group with $|G| = p^a m$ for p prime, $a \geq 1$, and $\gcd(p,m)=1$. Define $Syl_p(G) = \{P \leq G \text{ with } |P| = p^a\}$ and $n_p = |Syl_p(G)|$.

Then

1. $Syl_p(G) \neq \emptyset$
2. Any $P_1, P_2 \in Syl_p(G)$ are conjugate. There is a normal $P \in Syl_p(G) \Leftrightarrow n_p = 1$.
3. $n_p = [G : N_G(P)]$ and $n_p \mid m$ with $n_p \equiv 1 \bmod p$.
4. Any $H \leq G$ with $|H| = p^b$ is a subgp of some $P \in Syl_p(G)$.

[Very powerful tool, see HW/texts for applications.]

Proof of ①: Will show that if $m > 1$, there is an $H \leq G$ with $|H| = p^a \cdot m'$ with $1 \leq m' < m$. ($\Longleftrightarrow H \neq G$, and $[G:H]$ coprime to p). Suffices since repeating eventually results in $m' = 1$, i.e. $H \in Syl_p(G)$.

<u>Claim</u>: $G \curvearrowright X$ where (a) $|X|$ is coprime to p.
and (b) there are no $\underbrace{\text{orbits of size 1}}$. also called <u>fixed points</u>.

If so, $|X| = \sum |\mathcal{O}_i| = \sum [G : G_{x_i}]$ where $x_i \in \mathcal{O}_i$.

As $|X|$ is coprime to p, some $[G : G_{x_i}]$ is coprime to p. As $[G : G_{x_i}] > 1$, we take $H = G_{x_i}$.

<u>Pf of Claim</u>: $X = \{ S \subseteq G \mid |S| = p^a \}$ [S just a subset!]
Now $G \curvearrowright X$ by $g \cdot S = \{ g \cdot s \mid s \in S \}$.
Have $|X| = \binom{p^a m}{p^a} \equiv m \bmod p$ by #1 on today's HW.
As $m \not\equiv 0 \bmod p$, get (a). If $G \cdot S$ has size 1, then $g \cdot S = S$ for all $g \in G$. For $s_0 \in S$, have $(g s_0^{-1}) \cdot s_0 = g$ for any $g \Rightarrow g \in S$. But then $|S| = |G|$, a contractriction as $p^a < p^a m$. $\blacksquare$

Given $g \in G$, get $\mathrm{conj}_g \in \mathrm{Aut}(G)$. Any autom. takes subgps to subgroups preserving order/index. So G acts on $\mathrm{Syl}_p(G)$, and (2) is the claim there is only one orbit.

Lemma: G any gp with $H, K \leq G$. Then $\exists x \in G$ with $K \subseteq xHx^{-1} \iff$ the left action of K on G/H has a fixed point.

Pf [Skip!] $K \subseteq xHx^{-1} \iff Kx \subseteq xH \iff$ $\forall h \in H$ have $Kxh \subseteq x\underbrace{Hh}_{=H} \iff KxH \subseteq xH$, i.e. K fixes xH in G/H. ▨

Thm ($\Rightarrow$ Sylows Thms 2 and 4)
If $P \in Syl_p(G)$ and $Q \leq G$ is a p-group, then $\exists g \in G$ with $gQg^{-1} \subseteq P$.

Pf: To show: $Q \curvearrowright G/P$ has a fixed pt. Now

$$|G/P| = \sum |\mathcal{O}_i| = \#\text{fixed pts} + \overbrace{\sum_{|\mathcal{O}_i|>1} |\mathcal{O}_i|}^{\text{divides } |Q|}$$

$$\equiv \#\text{fixed pts} \mod p.$$

As $|G/P| \not\equiv 0 \mod p$ we have a fixed point and the lemma applies. ▨

For Sylow ③, have

$$N_G(P) := \{ g \in G \mid gPg^{-1} = P \}$$

which is the stab of P under the conj action on $\mathrm{Syl}_p(G)$, giving $n_p = [G : N_G(P)]$ from ② and the orbit-stab thm. As $P \leq N_G(P)$, get $n_p \mid m$. For $n_p \equiv 1 \bmod m$ see [Rezk §34].

<u>Cor</u>: Suppose $p < q$ are primes with $q \not\equiv 1 \bmod p$. Then any G of order $p \cdot q$ is cyclic.

<u>Pf</u>: Sylow ③ gives that $n_p = n_q = 1$. Let P and Q be the unique Sylow subgps, both of which are normal. Consider

$$K : P \longrightarrow \mathrm{Aut}(Q)$$
$$x \longmapsto \mathrm{conj}_x$$

Now $Q \cong C_q \Rightarrow \mathrm{Aut}(Q) \cong (\mathbb{Z}/q\mathbb{Z})^\times$ which has order $q-1$. Now $P = \langle x \rangle$ where $|x| = p$ so as $p \nmid q-1$ we have $|K(x)| = 1 \Rightarrow \mathrm{conj}_x = \mathrm{id}_Q$. So x commutes with y that gens Q. Set $z = xy$. If $z^k = e$, have $x^k = y^{-k} \in P \cap Q = \{e\}$ $\Rightarrow p \mid k$ and $q \mid -k$. So $|z| = pq$ and G is cyclic. ▨