Lecture 5: Group actions and the Orbit-Stabilizer Thm

§4.1 of [DF]
§18-21 of [R]

Previously on Math 500...

A (left) action of a group G on a set X is a map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ where ① $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$

② $e \cdot x = x$ for all $x \in X$.

Write $G \curvearrowright X$. Ex: $GL_n\mathbb{F}$ acts on $\mathbb{F}_n$ by mat. mult.

Ex: G acts on itself by ⓐ left-mult ⓑ conjugation ⓒ right-mult ($g \cdot x := x g^{-1}$)

When $G \curvearrowright X$, a $g \in G$ gives $\phi_g : X \to X$ by $\phi_g(x) := g \cdot x$. [For $GL_n\mathbb{F} \curvearrowright \mathbb{F}^n$, ϕ_A is the lin. trans assoc to A.]

Note that $\phi_g \circ \phi_h = \phi_{gh}$ as $\phi_g(\phi_h(x)) = g \cdot (h \cdot x) = (gh) \cdot x$. So each ϕ_g is a bijection since $\phi_{g^{-1}} \circ \phi_g = \phi_e(x) = id_X = \phi_g \circ \phi_{g^{-1}}$. So each $\phi_g \in Sym(X)$. Combining the above, get:

<u>Prop</u>: If $G \curvearrowright X$, then $G \longrightarrow \mathrm{Sym}(X)$, $g \longmapsto \phi_g$ is a homom.

<u>Prop</u>: If $\rho: G \to \mathrm{Sym}(X)$ is a homom, then $g \cdot x := (\rho(g))(x)$ is a group action

<u>Pf</u>: Exercise.

Suppose $G \curvearrowright X$. The <u>stabilizer</u> of $x \in X$ is $G_x := \{ g \in G \mid g \cdot x = x \}$

The <u>orbit</u> of $x \in X$ is $G \cdot x = \{ g \cdot x \mid g \in G \}$

<u>Ex</u>: $S_n \curvearrowright \{1, 2, \ldots, n\}$ $\quad (S_n)_k = H_k$ from HW 1.

$S_n \cdot 1 = \{1, 2, \ldots, n\}$

When only one orbit, the action is <u>transitive</u>

<u>Ex</u>: $G = GL_2\mathbb{R}$, $X = \mathbb{R}^2$

$G_{\binom{0}{0}} = G$

$G_{\binom{1}{0}} = \left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \,\middle|\, b \in \mathbb{R}^\times,\ a \in \mathbb{R} \right\}$

$G \cdot \binom{0}{0} = \left\{ \binom{0}{0} \right\}$

$G \cdot \binom{1}{0} = \mathbb{R}^2 \setminus \left\{ \binom{0}{0} \right\}$.

Prop: $G \curvearrowright X$. Suppose $x, y \in X$

① If $y = gx$, then $G_y = g G_x g^{-1}$

② Either $G\cdot x = G \cdot y$ or they are disjoint.

Pf [Skip if running low on time, as seems likely.]

① If $h \in G_x$, then $(ghg^{-1}) \cdot y = (gh) \cdot (g^{-1} \cdot (g \cdot x))$
$= g \cdot (h \cdot x) = g \cdot x = y$. So $g G_x g^{-1} \subseteq G_y$.

Same arg. shows $g^{-1} G_y g \subseteq G_x$ as $x = g^{-1} \cdot y$
and so $G_y \subseteq g G_x g^{-1}$. Hence $g G_x g^{-1} = G_y$.

② If $G \cdot x \cap G \cdot y$ contains some z, then
$z = g \cdot x$ and $z = h \cdot y$. Then $x = (g^{-1} h) \cdot y$
and each $g' \cdot x = (g' \cdot g^{-1} \cdot h) \cdot y \Rightarrow G \cdot x \subseteq G \cdot y$.
Reversing the roles of x and y gives $G \cdot x = G \cdot y$. ∎

Orbit-Stabilizer Thm: Suppose $G \curvearrowright X$ and $x \in X$.
Then there is a bijection $G/G_x \xrightarrow[\psi]{} G \cdot x$
given by $g G_x \longmapsto g \cdot x$.

Cor: $|G \cdot x| = [G : G_x]$

<u>Pf</u>: Set $H = G_x$ and $\mathcal{O} = G \cdot x$. The map $\psi : G/_H \longrightarrow \mathcal{O}$ is well-defined as if $gH = g'H$ then $g' = gh$ and $g' \cdot x = g \cdot (h \cdot x) = g \cdot x = x$. Surjectivity is clear.

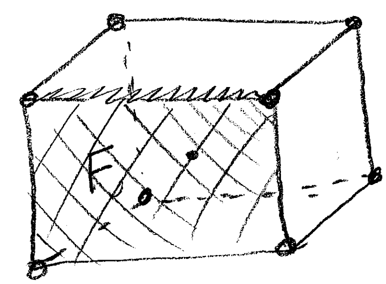
For injectivity, if $gH, g'H$ have $g \cdot x = g' \cdot x$ then $(g^{-1}g') \cdot x = x \Longrightarrow g^{-1}g' \in H \Rightarrow gH = g'H$

$\square$

<u>Ex</u>: C = cube in $\mathbb{R}^3$ with vertices $(\pm 1, \pm 1, \pm 1)$

$\tilde{G} = \{A \in GL_3\mathbb{R}$ that pres. the cube.$\}$

$G = \{A \in \tilde{G} \mid \det A > 0\}$.



$G \curvearrowright$ Faces of C with one orbit. So

$$6 = |G \cdot F| = [G : G_F]$$

So $|G| = [G : G_F] \cdot |G_F| = 6 \cdot 4 = 24$.

So $|\tilde{G}| = [\tilde{G} : G] \cdot |G| = 48$.

[Can repeat with edges and vertices.]