Lecture 3: Free groups

[Motivation: Group presentations, key example of infinite groups, universal properties.]

Consider $W = \{\text{finite strings in } a, b, a^*, b^*\}$

$= \{\underbrace{abba^*b^*bba^*}_{\text{word}},\ a^*ba^*ab,\ abab^*,\ e \ldots\}$ ← empty string

A word w is reduced when it does not contain adjacent "bad" pairs $\{aa^*, a^*a, bb^*, b^*b\}$.

Set $F(a,b) = \{w \in W \mid w \text{ reduced}\}$ ← Free group on a, b

Binary operation: concatenate, then "cancel" bad pairs

$u = aaba^*$ $\qquad aab\boxed{a^*a}b^*b^* \rightsquigarrow aa\boxed{bb^*}b^* \rightsquigarrow aab^*$

$v = ab^*b^*$

$u \circ v = aab^*$.

$v \circ u = ab^*b^*aaba^*$

Claim: This is a group.

ident: e.

inverses: $a^{-1} = a^*$ since $aa^* = e = a^*a$.

More generally "reverse and $*$"

$u^{-1} = (aaba^*)^{-1} = a\,b^*a^*a^*$

assocativity: Less clear because e.g. take

$w = ba^*b$ and

$$(u\cdot v)\cdot w = (aab^*)\cdot(ba^*b) = ab$$

$$u\cdot(v\cdot w) = u\cdot(ab^*b^* \cdot ba^*b)$$

$$= (aaba^*)\cdot(ab^*a^*b) = ab$$

See text or Rezk's notes for proof.

Note: Usually write a^{-1} for a^*, b^{-1} for b^* and abreviate $aabbba^{-1}a^{-1}$ as $a^2b^3a^{-2}$.

Q: Is $F(a,b)$ abelian? A. No as $ab \neq ba$.

Moral: $F(a,b)$ is the "largest possible group generated by two elts."

Thm: Suppose a group G is gen by x and y, i.e. $G = \langle x, y\rangle$. Then there exists a unique homomorphism $\phi : F(a,b) \longrightarrow G$ such that $\phi(a) = x$ and $\phi(b) = y$.

Cor: $G \cong F(a,b)/\mathrm{Ker}(\phi)$ [since $G = \langle x,y\rangle \Rightarrow \mathrm{im}(\phi) = G$.]

Pf: For a reduced $w = s_1 s_2 \cdots s_k \in F(a,b)$ with $s_i \in \{a, a^{-1}, b, b^{-1}\}$, set $\phi(w) = \phi(s_1)\phi(s_2)\cdots\phi(s_k)$ where $\phi(a) = x$, $\phi(a^{-1}) = x^{-1}$, $\phi(b) = y$, $\phi(b^{-1}) = y^{-1}$.

Why a homomorphism? Motivating example:

$$\phi(u \cdot v) = \phi(a a b^{-1}) = \phi(a)\phi(a)\phi(b^{-1}) = xxy^{-1}$$

$$\phi(u)\circ\phi(v) = (x\,x\,y\,x^{-1})(x\,y^{-1}\,y^{-1}) = xxy^{-1}$$

Details: If $w = u \cdot v$ where there's no cancelation when computing $u \cdot v$, then $\phi(w) = \phi(u)\phi(v)$.
Also $\phi(w^{-1}) = \phi(w)^{-1}$ for all $w \in F(a,b)$.
Any u, v can be written $u = u'r$, $v = r^{-1}v'$ so the mult $u' \cdot r$, $r^{-1} \cdot v'$, and $u' \cdot v'$ have no cancel. Then $\phi(u \cdot v) = \phi(u' \cdot v') = \phi(u')\phi(v')$ and $\phi(u)\cdot\phi(v) = \phi(u')\cdot \underbrace{\phi(r)\cdot\phi(r^{-1})}_{=e} \cdot \phi(v') =$.

∎

Note: Can replace $\{a,b\}$ with any set S and construct $F(S)$. Ex: $S = \{x, y, z\}$

Consider {finite strings in x, y, z, x^*, y^*, z^*}
set $F(a,b,c) = \{$all reduced words$\}$ ...

Given a set S, a <u>free group</u> on S is a group F with a function $i : S \to F$ such that for all groups G and functions $\phi : S \to G$ there exists a unique homom. $\Phi : F \to G$ such that

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\phi"'] & F \arrow[d, dashed, "\Phi"] \\ & G \end{tikzcd}$$

commutes, i.e. $\Phi \circ i = \phi$.

universal property

<u>Ex</u>: $S = \{a, b\}$ $F = F(a,b)$

$$\begin{array}{ccc} S & \xrightarrow{i} & F \\ a & \longmapsto & a \\ b & \longmapsto & b \end{array}$$

[Follows from the proof of thm]

<u>Ex</u>: $S = \{a, b\}$ $F = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle$ inside $SL_2\mathbb{R}$

(secret reason: hyperbolic geom)

<u>Note</u>: Free group on S is uniq up to isomorphism. If $i' : S \to F'$ is another free group

Consider

$$\begin{tikzcd} & F \arrow[d, dashed, "\phi"] \\ S \arrow[ur, "i"] \arrow[r, "i'"'] & F' \end{tikzcd} \qquad \begin{tikzcd} & F' \arrow[d, dashed, "\psi"] \\ S \arrow[ur, "i'"] \arrow[r, "i"'] & F \end{tikzcd}$$

<u>Claim</u>: ϕ and ψ are inverse bijections, hence group isomorphisms.

<u>Pf</u>: Have

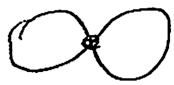
$$\begin{tikzcd} & F \arrow[dd, "\psi\circ\phi"] \\ S \arrow[ur, "i"] \arrow[dr, "i"'] \arrow[r, phantom, "\circlearrowleft"] & \\ & F \end{tikzcd}$$

← must be id_F by uniqueness in free group def.

Similarly, $\phi \circ \psi = id_{F'}$ as needed. ∎

If time remains, discuss

① Other "universal constructions" (free modules, tensor products, ...)

② π_1()