# Lecture 13: Solvable and nilpotent groups.

§49-51 of [RI]
§6.1 of [DF]

Last time: A composition series for a gp $G$ is a chain

$$\{e\} = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_{r-1} \trianglelefteq M_r = G$$

with all $M_K / M_{K-1}$ simple.

Thm: Exist for any finite $G$, composition factors unique up to permutation.

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Note: Some infinite $G$ have no comp. series, e.g. $\mathbb{Z}$.

A gp. $G$ is solvable when it has a finite chain

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_s = G \text{ with all } G_K/G_{K-1} \text{ abelian.}$$

Ex: $G$ abelian!

Ex: $D_{2n}$ via $\{1\} \trianglelefteq C_n \trianglelefteq D_{2n}$.

Ex: $Q_8$, $S_4$, $GL_2 \mathbb{F}_3$.

Non Ex: Any nonabelian finite simple group, e.g. $A_n$ for $n \geq 5$.

Thm: A finite $G$ is solvable $\iff$ all composition factors are cylic.

Feit-Thompson Thm: Any $G$ of odd order is solvable. (2)

(early 1960s; starting point for class of simple gps.)

[Heading to another char. of solvable gps.]

The commutator of $g, h \in G$ is $[g,h] := ghg^{-1}h^{-1}$.
$[= e \Leftrightarrow g$ and $h$ commute.] For $S, T \leq G$, set

$$[S,T] = \langle [s,t],\ s \in S,\ t \in T \rangle \leq G.$$

The commutator subgp of $G$ is $[G,G]$; it's normal since $g[x,y]g^{-1} = [gxg^{-1}, gyg^{-1}]$. In fact, its characteristic (inv. under $\mathrm{Aut}(G)$). The abelianization of $G$ is $ab(G) := G/[G,G]$. (largest poss. abelian quot.)

Ex: $G = F(a,b)$ $\quad [G,G] = \{aba^{-1}b^{-1}, \dots\}$

$$ab(G) \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z}^2.$$

Derived series:

$$G^{(0)} = G$$
$$G^{(1)} = [G,G]$$
$$G^{(k)} = [G^{(k-1)}, G^{(k-1)}]$$

a decending chain:

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq G^{(3)} \dots$$

Ex: For abelian $G$, $G^{(k)} = \{e\}$ for $k \geq 1$.

Ex: For nonabelian simple $G$, $G^{(k)} = G$ for all $k$.

Ex: For $F(a,b)$, $G^{(k)} \supsetneq G^{(k+1)}$ for all $k$.

Thm: $G$ is solvable $\iff$ $G^{(s)} = \{e\}$ for some $s$.

Pf. ($\Leftarrow$) As $G^{(k)}/G^{(k+1)} = G^{(k)}/[G^{(k)}, G^{(k)}]$ is abelian, then $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(s)} = \{e\}$ shows $G$ is solvable.

($\Rightarrow$) [Skip!] Suppose

$G = H_0 \trianglerighteq H_1 \trianglerighteq \cdots \trianglerighteq H_s = \{e\}$ with $H_k/H_{k+1}$ abelian.

abelian. As $H_0/H_1$ is abelian, $H_1 \supseteq [G,G] = G^{(1)}$

Similarly, $H_1/H_2$ is abelian $\Rightarrow H_2 \supseteq [H_1, H_1] \supseteq [G^{(1)}, G^{(1)}] = G^{(2)}$.

Repeating, see $H_k \supseteq G^{(k)}$ and so $G^{(s)} = \{e\}$ as needed. ▨

Cor: If $G$ is solvable, so is any subgp or quotient of $G$.

Pf: If $H \leq G$, have $\underbrace{[H,H]}_{H^{(1)}} \subseteq \underbrace{[G,G]}_{G^{(1)}}$ and more gen $H^{(k)} \subseteq G^{(k)}$ for all $k$.

So if $G^{(s)} = \{e\}$ so does $H^{(s)}$.

If $N \trianglelefteq G$, show $\left(G/N\right)^{(k)} = G^{(k)}N/N$

see §49 of [R1] for details. ▨

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Upper central series:

$$Z_0(G) = \{e\}$$

$$Z_1(G) = Z(G) = \{ g \in G \mid gx = xg \quad \forall x \in G \}$$

$$Z_{k+1}(G) = \pi^{-1}(Z(Q_k)) \text{ where}$$

$$\pi : G \to G/Z_k(G) =: Q_k$$

Have $\{e\} = Z_0(G) \trianglelefteq Z_1(G) \trianglelefteq Z_2(G) \trianglelefteq \cdots \leq G$

$G$ is <u>nilpotent</u> when some $Z_k(G) = G$.

<u>Ex:</u> Abelian gps are nilpotent as $Z_1(G) = G$.

<u>Ex:</u> Any p-group $G$ is nilpotent.

<u>Pf.</u> $Z_1(G) \neq \{e\}$ for any nontrivial p-group.

If $Z_1(G) \neq G$, then $Q_1 = G/Z_1(G)$ is a nontriv. p-group $\Rightarrow Z(Q_1) \neq \{e\} \Rightarrow Z_1(G) \subsetneq Z_2(G)$.

Repeating gives the claim. ▨

<u>Thm</u>: $G$ finite gp. Then $G$ is nilpotent

$\iff$ all Sylow subgps are normal

$\iff G = \prod_{p \mid |G|} P_p$ where $P_p \in Syl_p(G)$.

<u>Summary</u>:

$$\begin{pmatrix}\text{abelian}\\ \text{groups}\end{pmatrix} \subseteq \begin{pmatrix}\text{nilpotent}\\ \text{groups}\end{pmatrix} \subseteq \begin{pmatrix}\text{Solvable}\\ \text{groups}\end{pmatrix}$$

$$\begin{matrix}\cup\!\mid \\ \text{p-groups}\end{matrix}$$

since $Z_k(G) / Z_{k+1}(G) \cong \underbrace{Z}_{\text{abelian}}(Q_k)$