Lecture 11: Classification of finitely-generated abelian groups.

§5.2 of [DF]
§43-45 of [R1].

Last time: Thm: If G is a f.g. abelian group, then every subgp is also f.g..

Thm: G abelian. Then $G_{tors} = \{ g \in G \mid \underbrace{|g| < \infty}_{g \text{ is torsion}} \}$ is a subgp of G.

Thm: G f.g. abelian. If G is torsion, then $|G| < \infty$.

Today, write abelian groups additively, e.g. $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n$ instead of C_n and $\mathbb{Z}$ for C_∞.

Set $\mathbb{Z}^r = \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{r \text{ times}}$, which is torsion-free.

Also called the free-abelian group of rank r.

Universal prop: Set $e_i = (0, \ldots, \overset{i^{th} \text{ pos.}}{1}, \ldots, 0) \in \mathbb{Z}^r$. For all abelian G and elts $g_1, \ldots, g_r \in G$, there exists a unique homom. $\phi : \mathbb{Z}^r \longrightarrow G$ with $\phi(e_i) = g_i$

Pf: Exercise.

Thm: Every f.g. abelian group G is isomorphic to a unique gp of the form $\mathbb{Z}^r \times \mathbb{Z}/n_1 \times \mathbb{Z}/n_2 \times \cdots \times \mathbb{Z}/n_s$

where $r \geq 0$, $s \geq 0$, $n_i \geq 2$ and $n_i \mid n_{i+1}$ for all i.

r ← free rank; n_i ← invariant factors.

Ex: $\mathbb{Z}/3 \times \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}/9 \cong \mathbb{Z} \times \mathbb{Z}/3 \times \mathbb{Z}/18$

Reason: $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$ has order

$\text{lcm}(|g_i|)$. So $(1,1) \in \mathbb{Z}/2 \times \mathbb{Z}/9$ has order 18

$\Rightarrow \cong \mathbb{Z}/18$.

Note: Formula for $|(g_1, \ldots, g_n)|$ works even when G_i are not abelian.

Lemma: $\mathbb{Z}/a_1 \times \cdots \times \mathbb{Z}/a_n \cong \mathbb{Z}/a_1 a_2 \cdots a_n$

$\iff \gcd(a_i, a_j) = 1$ for all $i \neq j$.

Pf: § 43 of [R].

Ex: $\mathbb{Z}/3 \times \mathbb{Z}/9 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \cong \mathbb{Z}/15 \times \mathbb{Z}/225$

Pf of Thm: Will do as part of the study of modules over P.I.D. rings.

<u>Lemma</u>: p prime. Any f.g. abelian p-group is isom to unique $\mathbb{Z}/p^{a_1} \times \mathbb{Z}/p^{a_2} \times \cdots \times \mathbb{Z}/p^{a_\ell}$ where

$$1 \le a_1 \le a_2 \le \cdots \le a_\ell.$$

<u>Alt. Thm</u>: Any f.g. abelian gp is isom to a unique $\mathbb{Z}^r \times P_{p_1} \times \cdots \times P_{p_n}$ where $p_1 < \cdots < p_n$ are primes and each P_{p_i} is as in the lemma.

<u>Ex</u>: $\mathbb{Z}/6 \times \mathbb{Z}/30 \times \mathbb{Z}/60 \times \mathbb{Z}/300$ ← <u>invariant factor decomposition</u>

$$\cong (\mathbb{Z}/2)^2 \times (\mathbb{Z}/4)^2 \times (\mathbb{Z}/3)^4 \times (\mathbb{Z}/5)^2 \times (\mathbb{Z}/25)$$

↑ <u>primary decomposition</u>

<u>Group extensions</u>: G is an <u>extension</u> of K by H when $\exists\, H' \trianglelefteq G$ with $H' \cong H$ and $G/H \cong K$.

<u>Ex</u>: $H \times K$ is the trivial extention of K by H.

<u>Ex</u>: D_{2n} is an extension of C_2 by C_n.

An extension is <u>split</u> when $\exists\, K' \le G$ so that $K' \longrightarrow G/H'$ is an isomorphism.

<u>Ex</u>: C_4 and $C_2 \times C_2$ are both extensions of C_2 by C_2. For $G = \langle a \mid a^4 \rangle$ we take

$H' = \langle a^2 \rangle$ as $G/H' \cong C_2$. This extension is not split since only pos. for $K' = \langle a^2 \rangle \underset{\not\cong}{\longrightarrow} \{e_{G/H'}\}$.

Usually think of an extension as a pair of hom:

$$\begin{tikzcd} H \arrow[r, hook, "j"'] & G \arrow[r, two heads, "p"] & K \arrow[l, dashed, bend left, "s"] \end{tikzcd} \quad \text{where } j(H) = \ker(p).$$

This is split when $\exists\, s : K \to G$ with $p \circ s = id_K$.

<u>Q</u>: Given H, K can we classify all extensions of K by H?

<u>Next time:</u> Will do the split case.