Math 500: Final the Ultimate

Date/Time/Location: Thursday, December 14 from 8–11am in our usual classroom.

Office hours: My remaining office hours are:

- Wednesday, December 6 from 1:30–2:30pm.
- Thursday, December 7 at 2:00–3:00pm.
- Monday, December 11 from 10–11am.
- Tuesday, December 12 from 1–2pm.
- Wednesday, December 13 from 1:30–3:00pm.

Test format: There will be roughly twice as many questions as on one of the midterms. Most of the questions will be of similar difficulty to those on the second midterm, but a few many be more involved and/or difficult. I will not ask you to repeat proofs of theorems covered in class or in one of the texts.

Cheat sheet: The text will be closed book, but you may bring **three** sheets of standard US letter or A4 paper to the exam, on which you have written, printed, or copied anything that you think will be helpful. You can use both sides of each sheet, but must be able to read it without a magnifying glass or suchlike aide.

Material covered: The final will be comprehensive. The material on fields and Galois theory will make up about half the exam, with groups and rings each being about one quarter of the exam. Fair game is anything from:

- Any lecture.
- Part 1 of Rezk's notes.
- Part 2 of Rezk's notes, except for Sections 10, 24, 34, 47, 53–55.
- The following parts of Dummit and Foote: Chapters 1–6, with the exception of 6.2, Sections 7.1–7.5, Chapter 8, Chapter 9 except for the part of Section 9.6 after Corollary 22, Sections 10.1–10.3, Chapter 12, Chapter 13 except for 13.3 and 13.5, and Sections 14.1–14.3 and 14.5–14.7.

By far the most important material is that which appeared in lecture or on the homework.

Study resources: The course webpage, which is `http://dunfield.info/500` contains scanned lecture notes and solutions to the HW problems. You can pick up your graded HW 12 at any of my offices hours the week of December 11.

Homework problems for material after HW 12: Five problems about the material covered in the last three lectures is on the back of this sheet. Solutions will be posted.

Practice exam: Unfortunately, there is no practice exam. You may find useful practice questions on old algebra comp exams.

`https://math.illinois.edu/academics/graduate-program/coursework-and-exams/study-comp-exams`

Here's what would have been on HW 13. Solutions can be found on the course webpage.

1. Let θ be a root of $f(x) = x^3 - 3x + 1$. Show that $K = \mathbb{Q}(\theta)$ is a splitting field for f and that $\mathrm{Gal}(K/\mathbb{Q}) \cong C_3$. Express the other two roots of f explicitly in the form $a + b\theta + c\theta^2$ for $a, b, c \in \mathbb{Q}$.
2. Let $f(x)$ be an irreducible polynomial of degree 4 in $\mathbb{Q}[x]$ with discriminant D. Let K denote the splitting field of $f(x)$, viewed as a subfield of the complex numbers $\mathbb{C}$.
   (a) Prove that $\mathbb{Q}(\sqrt{D}) \subset K$.
   (b) Let τ denote complex conjugation and τ_K its restriction to K. Prove that τ_K is an element of $\mathrm{Gal}(K/\mathbb{Q})$ of order 1 or 2 depending on whether every element of K is real or not.
   (c) Prove that if $D < 0$ then $\mathrm{Gal}(K/\mathbb{Q})$ cannot be isomorphic to C_4.
   (d) Prove generally that $\mathbb{Q}(\sqrt{D})$ for squarefree $D < 0$ is not a subfield of any Galois $L/\mathbb{Q}$ where $\mathrm{Gal}(L/\mathbb{Q}) \cong C_4$.
3. Consider $f(x) = (x^3 - 2)(x^3 - 3)$ in $\mathbb{Q}[x]$.
   (a) Determine the Galois group of $f(x)$ over $\mathbb{Q}$. That is, if K is the splitting field of f, compute $\mathrm{Gal}(K/\mathbb{Q})$. Note: You may assume that $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt[3]{3})$ are distinct subfields of K. (This can be verified by checking that $\sqrt[3]{3} - \sqrt[3]{2}$ is a root of $x^9 - 3x^6 + 165x^3 - 1$ and that the latter polynomial is irreducible.)
   (b) Find all subfields of K that contain $\mathbb{Q}(\zeta)$, where ζ is a primitive 3^{rd} root of unity.
4. Let $F \subset \mathbb{R}$ be a field. Let a be an element of F which has a real n^{th} root $\alpha = \sqrt[n]{a}$, and set $K = F(\alpha)$. Prove that if L is any Galois extension of F contained in K then $[L : F] \leq 2$. **Hint**: Let M be the Galois closure of K, concretely $M = F(\alpha, \zeta_n)$. Now compute $N_{M/L}(\alpha)$, where $N_{M/L}$ is the norm from Problem 8 on HW 12.

Credit: Problems from Dummit and Foote, Sections 14.6 and 14.7.