Math 500: HW 12 due Friday, December 1, 2023.

1. Let F be a field of characteristic $\neq 2$.

   (a) Suppose $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1, D_2, or D_1D_2 is a square in F. Prove that K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group.

   (b) Conversely, suppose K/F is a Galois extension with $\mathrm{Gal}(K/F)$ isomorphic to the Klein 4-group. Prove that $K = F(\sqrt{D_1}, \sqrt{D_2})$ for D_1 and D_2 in F where none of D_1, D_2, or D_1D_2 is a square in F.

2. Suppose K is the splitting field over $\mathbb{Q}$ of a cubic polynomial $f(x) \in \mathbb{Q}[x]$. Show that if $\mathrm{Gal}(K/\mathbb{Q})$ is the cyclic group of order 3, then all the roots of f are real. Hint: by calculus, f must have at least one real root.

3. Suppose $\mathbb{Q}(\alpha)/\mathbb{Q}$ is algebraic and let L be the splitting field of $m_{\alpha,\mathbb{Q}}(x)$. If p is a prime dividing the order of $\mathrm{Gal}(L/\mathbb{Q})$, show that there is a subfield F of L with $[L:F] = p$ and $L = F(\alpha)$.

4. Consider $K = \mathbb{Q}(\sqrt[8]{2}, i)$ from Example 3 of Section 14.2 of [DF], and let $F_1 = \mathbb{Q}(i)$, $F_2 = \mathbb{Q}(\sqrt{2})$, $F_3 = \mathbb{Q}(\sqrt{-2})$. Prove that $\mathrm{Gal}(K/F_1) \cong \mathbb{Z}/8$, $\mathrm{Gal}(K/F_2) \cong D_8$, $\mathrm{Gal}(K/F_3) \cong Q_8$. (Note: in particular, you need to show all these extensions are Galois extensions.)

5. Let $L \subseteq \mathbb{C}$ be a subfield such that $L/\mathbb{Q}$ be a finite Galois extension of degree 3, and let $\omega := e^{2\pi i/3} \in \mathbb{C}$. Show that if $\omega \notin L$, then $L(\omega)/\mathbb{Q}$ is a Galois extension with $\mathrm{Gal}(L(\omega)/\mathbb{Q}) \approx \mathbb{Z}/3 \times \mathbb{Z}/2 \approx \mathbb{Z}/6$.

6. Suppose $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial and L a splitting field for f. Suppose further than $G = \mathrm{Gal}(L/\mathbb{Q}) \approx S_4$. Let θ be a root of f and set $K = \mathbb{Q}(\theta)$. Prove that K is an extension of $\mathbb{Q}$ of degree 4 which has no proper subfields. Are there any Galois extensions of degree 4 over $\mathbb{Q}$ with no proper subfields? **(This problem has been corrected.)**

7. Find a primitive generator for $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ over $\mathbb{Q}$. Be sure to justify your answer.

8. Let K/F be a Galois extension and $\alpha \in K$. Define the *norm* of α from K to F to be

$$N_{K/F}(\alpha) = \prod_{\sigma \in \mathrm{Gal}(K/F)} \sigma(\alpha)$$

   (a) Prove that $N_{K/F}(\alpha) \in F$.

   (b) Prove that $N_{K/F}(\alpha\beta) = N_{K/F}(\alpha)N_{K/F}(\beta)$. Thus, the norm gives a group homomorphism $K^\times \to F^\times$.

   (c) Prove that $N_{K/F}(a\alpha) = a^n N_{K/F}(\alpha)$ where $a \in F$ and $n = [K:F]$.

   (d) Let $K = F(\sqrt{D})$ be a quadratic extension of a field F whose characteristic is not 2. Show that $N_{K/F}(a + b\sqrt{D}) = a^2 - Db^2$, where $a, b \in F$. **(This subproblem has been corrected.)**

   (e) Let K be the splitting field of $x^3 - 2$ over $\mathbb{Q}$. Compute $N_{K/Q}$ for $\alpha = \sqrt[3]{2}$ and for $\zeta = \zeta_3$.

   Note: It is possible to define $N_{K/F}(\alpha)$ even when K/F is not Galois, see Problem #17 in Section 14.2 of our text.

Credit: Problems by Charles Rezk or Dummit and Foote.