## Math 500: HW 9 due Friday, October 27, 2023.

**Webpage:** http://dunfield.info/500
**Office hours:** Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.

1. Let $R$ be a commutative ring with 1. Prove that $M$ and $\mathrm{Hom}_R(R, M)$ are isomorphic as $R$-modules.

2. (a) Suppose $M$ is a $\mathbb{Z}$-module. If $(M, +)$ has a nontrivial element $m$ of finite order, show that $M$ is not a free module on any subset $S$.

   (b) Prove or disprove: the same result holds with $\mathbb{Z}$ replaced by any field $\mathbb{F}$.

3. Suppose $R$ is an integral domain. Prove that any free $R$-module is torsion free.

4. Let $R$ be an integral domain and $M$ any non-principal ideal of $R$ such that $M \neq (0)$, considered as an $R$-module. Show that (i) $M$ is torsion free (ii) $\operatorname{rank} M = 1$, but (iii) $M$ is not a free module. (Hint: Given $f \in M$, when is $M/(Rf)$ a torsion module? See also problem 12.1.5 in [DF].)

5. Let $R$ be a PID with fraction field $F$, and note that $F$ has the structure of an $R$-module (by multiplication). Show that the rank of $F$ as an $R$-module is equal to 1. (Hint: think about the $R$-module $F/R$.)

6. Let $R$ be a PID, $B$ a torsion $R$-module, and $p$ a prime element in $R$. Prove that if $pb = 0$ for some $b \in B$, $b \neq 0$, then $\mathrm{Ann}(B) \subseteq (p)$. (Recall that $\mathrm{Ann}(B) = \{r \in R \mid rB = 0\}$.)

7. Let $R = \mathbb{Z}[i]$. Classify all finitely generated $R$-modules $M$ with the property that $5M = 0$.

8. Use the rational canonical form to determine all conjugacy classes in the group $G = \mathrm{GL}_2\mathbb{F}_3$. Specifically, give a list of matrices in rational canonical form, one in each conjugacy class. Warning: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ is *not* in rational canonical form, even though it's diagonal.

9. Find all similarity classes of $6 \times 6$ matrices over $\mathbb{Q}$ with minimal polynomial $(x+2)^2(x-1)$. (It suffices to give all lists of invariant factors and write out some of their corresponding matrices).

10. **This problem has been removed because we won't cover this material until Friday.**

    Compute the Jordan canonical form (over $\mathbb{C}$) of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$.