Math 500: HW 2 due Friday, September 1, 2023.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Show $G = SL_2(\mathbb{Z}/3)$ has $|G| = 24$ and give examples of subgroups of G of orders 3, 8, and 6. Hint: see [DF] §2.4 #9-10.

2. Show that none of the following groups are isomorphic to each other: $SL_2(\mathbb{Z}/3)$, S_4, D_{24}.

3. Show that every non-identity element of a free group has infinite order. (Hint: need to use the description of elements of free group in terms of reduced words.)

4. Give a presentation of S_4 with 2 generators, and prove that your presentation is correct.

5. Prove that $\langle a, b \mid a^2b^{-2}, aba^{-1}b \rangle$ is a presentation of the quaternion group of order 8.

6. Show that $\langle a, b \mid aba^{-1}b^{-2},\ bab^{-1}a^{-2} \rangle \cong \{e\}$.

7. Let S_3 act on the set Ω of ordered pairs (i, j) with $1 \le i, j \le 3$, by $\sigma((i, j)) := (\sigma(i), \sigma(j))$. Let $\phi \colon S_3 \to \mathrm{Sym}(\Omega)$ denote the corresponding homomorphism.

   (i) Find the orbits of this action.

   (ii) For each $\sigma \in S_3$, find the cycle decomposition of $\phi(\sigma) \in \mathrm{Sym}(\Omega)$.

   (iii) For each orbit $\mathcal{O}$ of S_3 acting on Ω, pick some $a \in \mathcal{O}$ and describe its stabilizer subgroup $\mathrm{Stab}(a) \subseteq S_3$.

8. Let G be a group and m a positive integer. Show that the following are equivalent.

   (a) G acts transitively on some set X of size m.

   (b) There exists a subgroup $H \le G$ with $|G : H| = m$.

9. Let G be a group with subgroup H, and let $\lambda \colon G \to \mathrm{Sym}(G/H)$ be the left-coset action, defined by $\lambda_g(xH) := gxH$. Show that $K := \operatorname{Ker} \lambda = \bigcap_{x \in G} xHx^{-1}$.

10. Let G be a finite group with a subgroup H of index m, and let $K := \operatorname{Ker} \lambda$ as in the previous problem.

    (a) Show that $|H : K|$ divides $(m-1)!$.

    (b) Use this to show that if p is the smallest prime which divides $|G|$, then any subgroup of index p in G is normal.

Credit: Problems 3–5, and 7 are from [DF], the rest from [R].