Math 500: HW 1 due Friday, August 25, 2023.

Webpage: http://dunfield.info/500
Office hours: Wednesdays 1:30–2:30pm and Thursdays 2:00–3:00pm; additional times possible by appointment.
Textbook: Dummit and Foote, *Abstract Algebra*, 3rd edition.
Suplemental text: Charles Rezk, Lecture Notes for Math 500, posted on our course webpage.

1. Let G be a group. Suppose $a, b \in G$ are such that $|a| = m$, $|b| = n$, and $ab = ba$. Show that if $\gcd(m, n) = 1$, then $c := ab$ has order mn. (Here $|a|$ is the order of a.)

2. Show that if an element a in a group G has finite order m, then for any positive integer k such that $\gcd(k, m) = 1$, there exists an element $x \in G$ such that $x^k = a$.

3. Prove that if G is a group such that $a^2 = e$ for all $a \in G$, then G is abelian.

4. Consider $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ in $G = \mathrm{GL}_2\mathbb{R}$. Compute the orders of elements A, B, AB, BA. Fun fact: The subgroup $\langle A, B\rangle$ of G turns out to be
$$\mathrm{SL}_2\mathbb{Z} := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \;\middle|\; a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

5. Show that for each $k \in \{1, \ldots, n\}$ and $n \geq 1$, the subgroup $H_k := \{\sigma \in S_n \mid \sigma(k) = k\}$ is isomorphic to S_{n-1}. Determine whether these subgroups are normal.

6. Give an example of subgroups $K \leq H \leq G$ such that K is normal in H, H is normal in G, but K is not normal in G. (Hence "is normal subgroup" is not a transitive relation.)

7. Prove that the multiplicative groups $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ are not isomorphic.

Credit: Problems 3 and 7 are from [DF], the rest from [R].