The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra has many proofs. Here are references to a few of my favorites. In place of the cancelled lecture on Friday, March 13 please read one of these to make your peace with the Fundamental Theorem of Algebra.

Proof by compactness. All you really need to prove the Fundamental Theorem of Algebra is the Extreme Value Theorem for functions of two variables, that is, that any continuous function on a compact set in $\mathbb{R}^2$ has a minimum. A particularly concise version of this is on the next two pages, taken from *The American Mathematical Monthly*, Vol. 74, No. 7 (1967), 854–855. It is basically the same as the proof by d'Alambert from 1746, though compactness was not rigorously understood at that point.

Proof by the Inverse Function Theorem. If you like the Inverse Function Theorem, it's easy to combine this with compactness to prove the Fundamental Theorem of Algebra, see pages 1–4 of my old notes from 2010.

Proof by Newton's method. A variant of the preceding proof is to directly establish a special case of the Inverse Function Theorem using Newton's method. See pages 1–8 of my old notes from 2010.

Proof by differential topology. The proof on pages 8-9 of Milnor's classic text *Topology from a differentiable viewpoint* is very nice, assuming just the basics about differentiating functions from $\mathbb{R}^n \to \mathbb{R}^m$.

Proof by complex analysis. The basic idea is to apply Liouville's theorem, i.e. that a bounded entire function is constant, to $1/p(z)$ where $p(z) \in \mathbb{C}[z]$ is a non-constant polynomial with (allegedly) no roots. This proof can be found in your favorite complex analysis text, or on Wikipedia.

Proof with minimal analysis and maximum algebra. Propositions 30 and 31 at the end of Section 14.6 of Dummit and Foote use Galois Theory to prove the FTA from two basic consequences of the Intermediate Value Theorem. You would want to wait until we have proved the Fundamental Theorem of Galois Theory before reading this version.

so that, for c as stated, w_k does converge to a root of (1).

Now let p be any nonconstant polynomial, z_0 any complex number such that $p'(z_0) \neq 0$. With the substitutions $z = z_0 + w$, $a = p(z_0) + p'(z_0)c$, the result just obtained shows that the equation $p(z) = a$ has a root for all values of a close enough to $p(z_0)$. In other words, if z_0 is not a zero of p', $p(z_0)$ is an interior point of S. This proves the Lemma.

THEOREM. *With the notation of the lemma, S is the set of all complex numbers.*

Proof. The complement of S is open (B), and $S-T$ is open by the lemma. But as T is finite (A) its complement cannot consist of two disjoint nonempty open sets (C). Hence the complement of S must be empty.

AN EASY PROOF OF THE FUNDAMENTAL THEOREM OF ALGEBRA

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THEOREM. *Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial. Then P has a zero.*

Proof. We shall show first that $|P(z)|$ attains a minimum as z varies over the entire complex plane, and next that if $|P(z_0)|$ is the minimum of $|P(z)|$, then $P(z_0) = 0$.

Since $|P(z)| = |z|^n |a_n + a_{n-1}/z + \cdots + a_0/z^n|$ $(z \neq 0)$, we can find an $M > 0$ so large that

(1) $$|P(z)| \geqq |a_0| \qquad (|z| > M).$$

Now, the continuous function $|P(z)|$ attains a minimum as z varies over the compact disc $\{|z| \, |z| \leqq M\}$. Suppose, then, that

(2) $$|P(z_0)| \leqq |P(z)| \qquad (|z| \leqq M).$$

In particular, $|P(z_0)| \leqq P(0) = |a_0|$ so that, by (1), $|P(z_0)| \leqq |P(z)|$ $(|z| > M)$. Comparing with (2), we have

(3) $$|P(z_0)| \leqq |P(z)| \qquad \text{(all complex } z\text{)}.$$

Since $P(z) = P((z-z_0)+z_0)$, we can write $P(z)$ as a sum of powers of $z-z_0$, so that for some complex polynomial Q,

(4) $$P(z) = Q(z - z_0).$$

By (3) and (4),

(5) $$|Q(0)| \leqq |Q(z)| \qquad \text{(all complex } z\text{)}.$$

We shall show that $Q(0) = 0$. This will establish the theorem since, by (4), $P(z_0) = Q(0)$.

Let j be the smallest nonzero exponent for which z^j has a nonzero coefficient in Q. Then we can write $Q(z) = c_0 + c_j z^j + \cdots + c_n z^n$ $(c_j \neq 0)$. Factoring

z^{j+1} from the higher terms of this expression, we have

$$Q(z) = c_0 + c_j z^j + z^{j+1} R(z), \tag{6}$$

where $c_j \neq 0$ and R is a complex polynomial.

If we set $-c_0/c_j = re^{i\theta}$, then the constant $z_1 = r^{1/j} e^{i\theta/j}$ satisfies

$$c_j z_1^j = -c_0. \tag{7}$$

Let $\epsilon > 0$ be arbitrary. Then, by (6),

$$Q(\epsilon z_1) = c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1). \tag{8}$$

Since polynomials are bounded on finite discs, we can find an $N > 0$ so large that, for $0 < \epsilon < 1$, $|R(\epsilon z_1)| \leqq N$. Then, by (7) and (8) we have, for $0 < \epsilon < 1$,

$$\begin{aligned} |Q(\epsilon z_1)| &\leqq |c_0 + c_j \epsilon^j z_1^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &\leqq |c_0 + \epsilon^j (c_j z_1^j)| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ &= |c_0 + \epsilon^j (-c_0)| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ &= (1 - \epsilon^j) |c_0| + \epsilon^{j+1} (|z_1|^{j+1} N) \\ &= |c_0| - \epsilon^j |c_0| + \epsilon^{j+1} (|z_1|^{j+1} N). \end{aligned} \tag{9}$$

If $|c_0| \neq 0$, then we can take ϵ so small that $\epsilon^{j+1}(|z_1|^{j+1} N) < \epsilon^j |c_0|$. In that case, by (9)

$$|Q(\epsilon z_1)| \leqq |c_0| - \epsilon^j |c_0| + \epsilon^{j+1}(|z_1|^{j+1} N) < |c_0| = |Q(0)|,$$

contradicting (5). So $|c_0| = 0$, and therefore $Q(0) = c_0 = 0$.

MATHEMATICAL EDUCATION NOTES

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SEARCHING FOR MATHEMATICAL TALENT IN WISCONSIN, III.

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The Wisconsin High School Mathematical Talent Search operated for its third year in 1965–66, with continuing financial support from the National Science Foundation. Reports for preceding years are given in references [1], [2].