Lecture 40:

Goal Thm: G a finite gp. Then $\exists$ a Galois extension $K/\mathbb{C}(t)$ with group G.

[Reminder: Big open prob when base field is $\mathbb{Q}$.]

Last time: Given an irreducible curve $V \subseteq \mathbb{C}^2$, and a non-constant poly $h \in \mathbb{C}[V]$ (e.g. corresp. to projection onto the x-axis), get that $K = \mathbb{C}(V)$ is a finite extension of $\mathbb{C}(t)$.

Plan: (1) Given G, find a curve V in $\mathbb{P}^n_{\mathbb{C}}$ on which G acts via symmetries, so that

$V/G = \mathbb{P}^1_{\mathbb{C}} =$ .

(2) Each $\sigma \in G$, thought of as a symm $\sigma : V \longrightarrow V$ gives an automorphism of $K = \mathbb{C}(V)$ via $\sigma^*(f) = f \circ \sigma^{-1}$ where $f \in K$ is viewed as a rat'l fn $f : V \longrightarrow \mathbb{P}^1_{\mathbb{C}}$

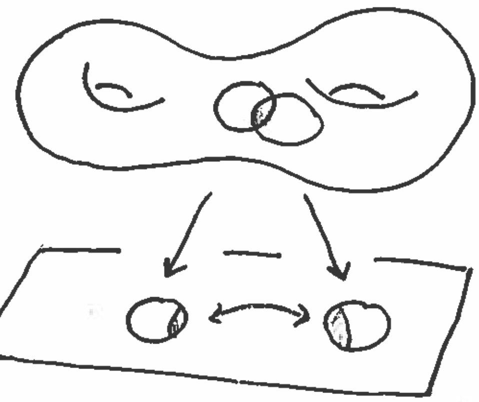
Aside: $\tau^*(\sigma^*(f)) = \tau^*(f \circ \sigma^{-1}) = (f \circ \sigma^{-1}) \circ \tau^{-1}$
$= (\tau \circ \sigma)^* f$

3 $K_G = \mathbb{C}(V)_G = \mathbb{C}(V/G) = \mathbb{C}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$

[In the time left, I can't do the whole proof as you need one more perspective.]

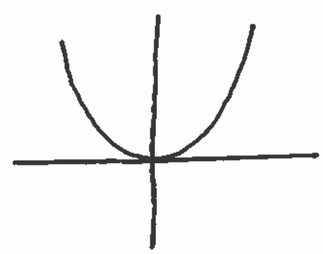
$$\left\{\begin{matrix}\text{finite extensions}\\ \text{over } \mathbb{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Alg. curves}\\ \text{in } \mathbb{P}^n_{\mathbb{C}}\end{matrix}\right\}$$

$$\text{Complex Analysis} \supseteq \left\{\begin{matrix}\text{Riemann}\\ \text{surfaces}\end{matrix}\right\}$$



Also, need some topology of covering spaces.

<u>Back to example</u>: $V = \mathbb{V}(x^2 - yz)$

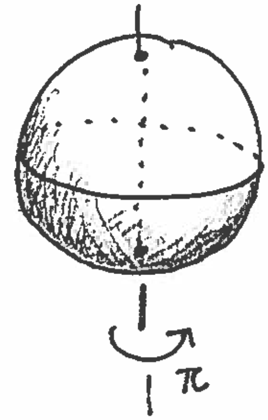


$G = C_2$ gen by $\sigma : x \longmapsto -x$

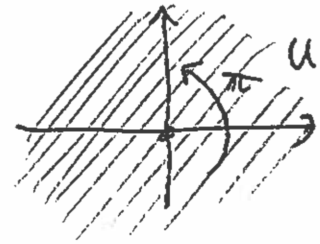
$V/G = y\text{-axis} = \mathbb{V}(x=0)$

Here, both V and V/G are $\cong \mathbb{P}^1_{\mathbb{C}}$

Geometrically, σ is



since projectively



$u \to -u$

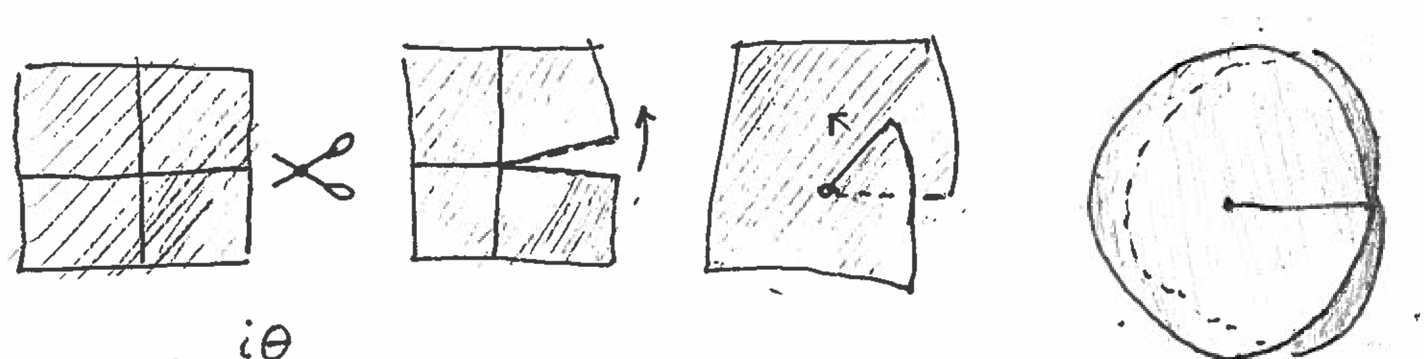
$\sigma(u:v) = (-u:v)$.

Well, this is clear on $\mathbb{C} = (u:1) \subseteq \mathbb{P}^1_{\mathbb{C}}$, but how do we know what it looks like near $\infty = (1:0)$? On $\mathbb{P}^1_{\mathbb{C}} \setminus \{(0:1)\} = \{(1:v)\}$ we have $\sigma(1:v) = (-1:v) = (1:-v)$.

What about the quotient map $V \longrightarrow V/G$?

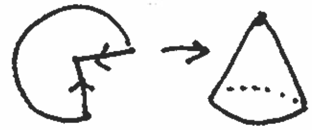
$$(u:v) \longrightarrow (u^2:v^2)$$

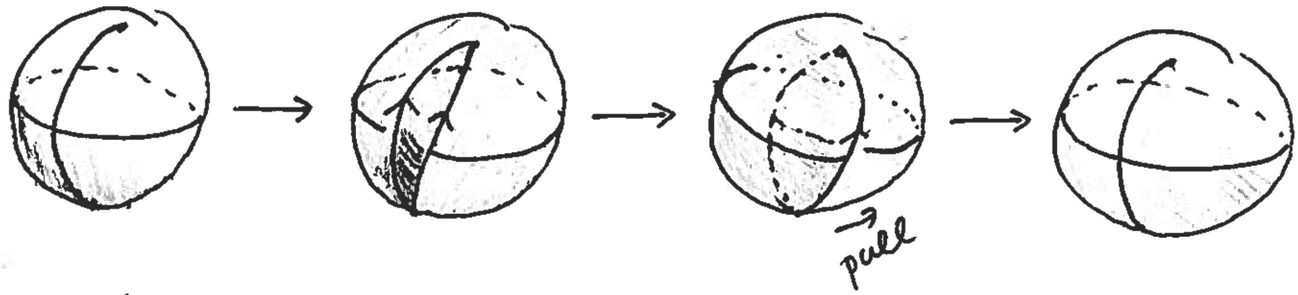
On $(u:1)$ this is $u \mapsto u^2$:



$$u = re^{i\theta} \longmapsto u^2 = r^2 e^{2i\theta}$$

Same for $(1:v)$: $v \longmapsto v^2$

This is like making a cone,  but there is "too much" angle around the cone pt instead of too little. On $\mathbb{P}^1_{\mathbb{C}}$, have



The map is 2-1 generically and 1-1 locally,

except near 0 and ∞.

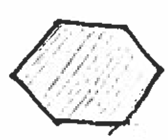
This is an example of a <u>branched cover</u> in topology.

A <u>group action</u> of G on set X is a map

$$G \times X \longrightarrow X \qquad \text{satisfying} \qquad 1 \cdot x = x \quad \text{for all } x \in X$$

$$(g, x) \longmapsto g \cdot x \qquad\qquad g \cdot (h \cdot x) = (gh) \cdot x.$$

<u>Ex:</u> S_n acts on $\{1, 2, \ldots, n\}$

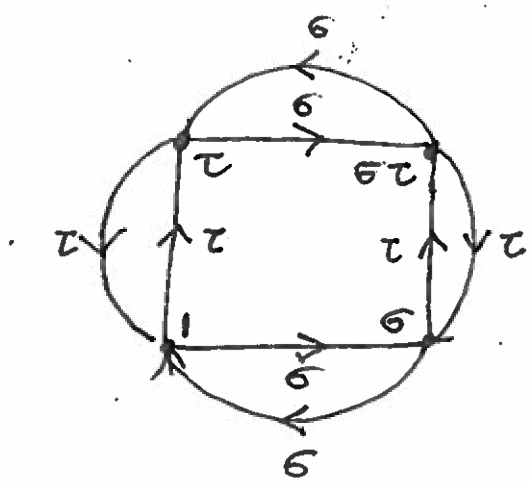
D_{2n} acts on an n-gon 

Given a group G, let's make it act on <u>some</u> geometric object.

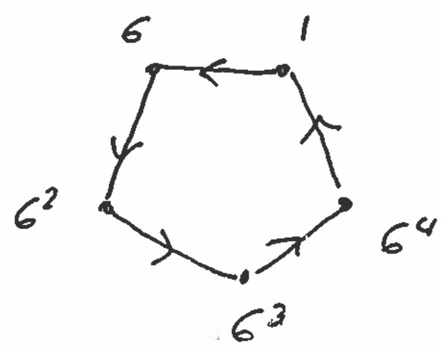
<u>Def</u>: Let S be a generating set for G. The <u>Cayley graph</u> $\Gamma(G, S)$ has

1. a vertex v_g for each $g \in G$.
2. an edge labled s from v_g to v_{gs} $\forall g \in G, s \in S$.

<u>Ex</u>: $G = C_2 \times C_2 = \{1, \tau, \sigma, \sigma\tau\}$ $S = \{\tau, \sigma\}$

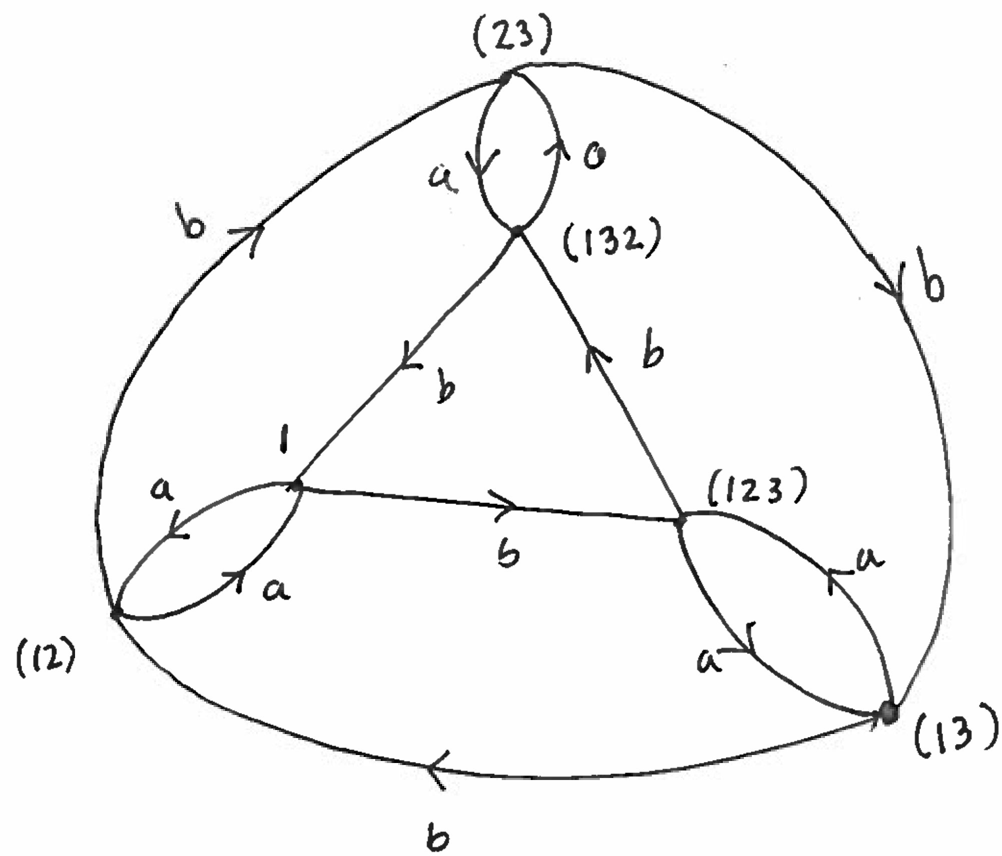


<u>Ex</u>: $G = C_n$, $S = \{\sigma\text{-gen}\}$



Ex: $S_3 = \{1, (12), (13), (23), (123), (132)\}$

$S = \{a = (12), \ b = (123)\}$

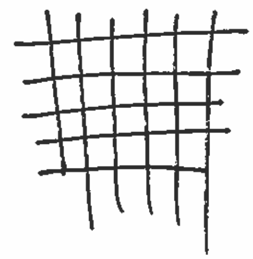


Q: What is $abab^{-1}ab$? A. $(12) = a$.

For any (G, S) the group G acts on $\Gamma(G,S)$ by $g \cdot v_h = v_{gh}$. This respects the edges,

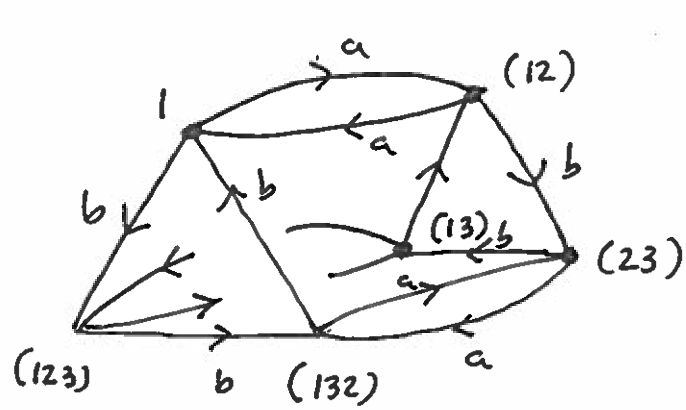
since an "s" edge joins $v_h \longmapsto v_{hs}$.

<u>Aside</u>: (a) Can do for infinite groups, leading to geometric group theory

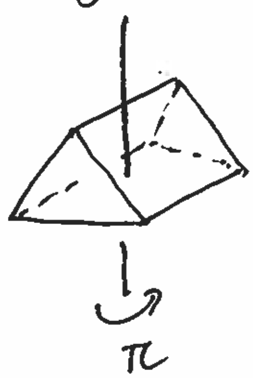


(b) Certain families of Cayley graphs are <u>expanders</u>

In the main example:



a acts by rotation by π



b acts by rotation by $2\pi/3$