Lecture 36: Curves over $\mathbb{C}$

Last time: Plane conics in $\mathbb{P}^2_{\mathbb{R}}$.

Have $\mathbb{V}_{\mathbb{R}^2}(x^2+y^2-1) =$ , but $\mathbb{V}_{\mathbb{C}^2}(x^2+y^2-1) \neq$ 

[Today, will see how projective space fixes this.]

Q: What is $\mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(x^2+y^2-z^2) = V$?

A. All non-degenerate real conics are the same in projective space, so consider instead

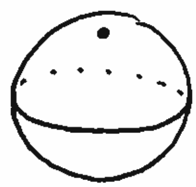
$$V' = \mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(x^2 - yz)$$

which is $P_A(V)$ for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix}$, as you can check.

Now
$$V' = \mathbb{V}_{\mathbb{C}^2}(x^2-y) \cup \{(0:1:0)\} \quad \leftarrow \text{at } \infty$$

param by x, so $\approx$ to $\mathbb{C}$.

and so

$$V' = \mathbb{C} \cup \{pt\} =$$ 

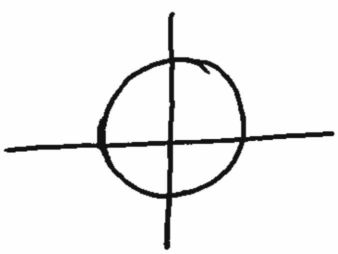
Explicitly, have $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\cong} V'$

$$(u:v) \longmapsto (uv : u^2 : v^2)$$

well-defined as all terms have the same degree

Let k be a field. An affine variety $V = \mathbb{V}(f) \subseteq k^2$ is <u>nonsingular</u> or <u>smooth</u> if $Df = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \neq 0$ at every point of V.

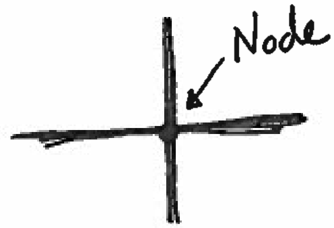
<u>Smooth</u>:



$\mathbb{V}(x^2+y^2-1)$

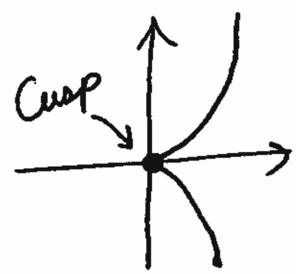
$Df = (2x, 2y)$

<u>Singular</u>:



$\mathbb{V}(xy)$

$Df = (y, x)$



$\mathbb{V}(y^2 - x^3)$

$Df = (-3x^2, 2y)$

When $k = \mathbb{R}$ or $\mathbb{C}$, the Implicit Function Theorem tells us that a non-singular variety in k^2 like this looks locally like k. Such varieties are called <u>curves</u>.

For $C = \mathbb{V}_{\mathbb{P}^2_k}(f)$, we say it is smooth if all three affine curves

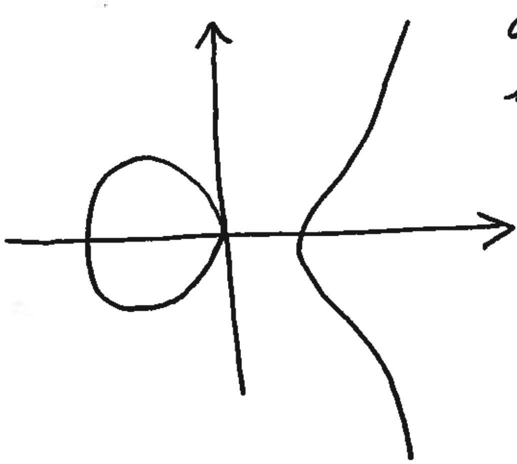
$$C \cap \{(x:y:1)\},\quad C \cap \{(1:y:z)\},\quad C \cap \{(x:1:z)\}$$

are smooth.

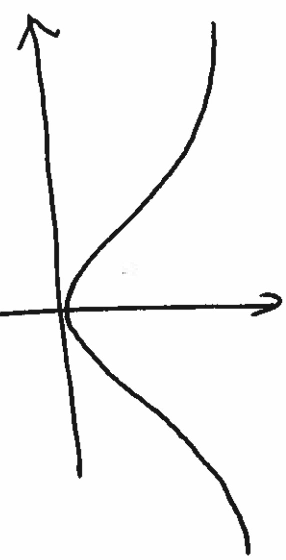
After lines and conics, the next examples are <u>elliptic curves</u>: $C = \mathbb{V}_{\mathbb{P}^2_k}\left(y^2 - x(x^2+ax+b)\right)$

[Aside: over $\mathbb{R}$ or $\mathbb{C}$, any curve coming from a cubic equation can be put into this form by a projective transformation.]

<u>Ex</u>: $k = \mathbb{R}$



In $\mathbb{P}^2_{\mathbb{R}}$, both also have a single point at ∞, namely $(0:1:0)$.



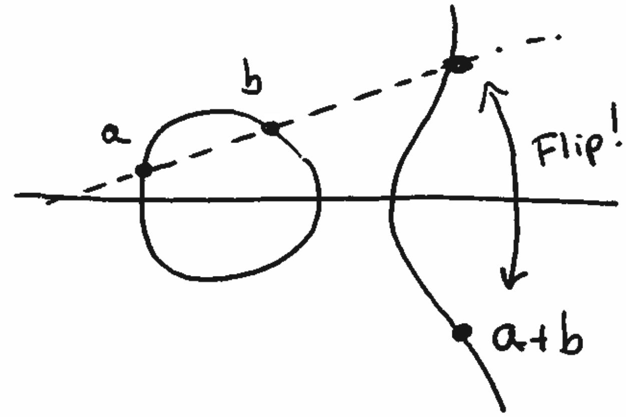
$y^2 = x(x-1)(x+1)$ $\qquad\qquad$ $y^2 = x(x^2+1)$

Suppose C is a <u>smooth</u> elliptic curve. Then we can make it into an abelian group:

ⓐ $\mathcal{O} = (0:1:0)$ is the ident. element.

ⓑ The inverse of (x,y) is $(x,-y)$

ⓒ If $a,b,c \in C$ lie on a line, then $a+b+c = \mathcal{O}$.



Q: Suppose $k=\mathbb{C}$. Is $C =$  as with a line or a conic?

Consider the map $\pi : C \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. On $C \cap \mathbb{C}^2$

$$(x:y:z) \longmapsto (x:z)$$

this is just the projection $(x,y) \longrightarrow x$,

and $(0:1:0)$ in C goes to the point at ∞ in $\mathbb{P}^1_{\mathbb{C}}$.

<u>Claim</u>: π is generically 2-to-1. If C is defined by

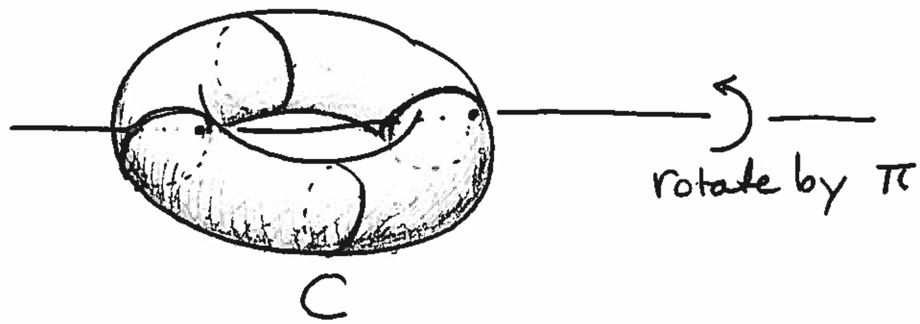
$$y^2 = x(x-\alpha)(x-\beta) \qquad (\star)$$

then $\pi^{-1}(p) =$ two pts except for $p \in \{0, \alpha, \beta, \infty\}$.

Point: For fixed x, ⊛ has two solutions unless the RHS vanishes, in which case there's only one.

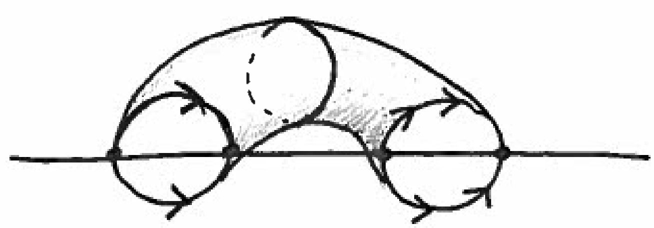
The symmetry of C given by $\overset{a}{\overset{\shortparallel}{(x,y)}} \longrightarrow \overset{-a}{\overset{\shortparallel}{(x,-y)}}$ ($a \longmapsto -a$) respects π, and so $C/_{a \sim -a} = \mathbb{P}^1_{\mathbb{C}}$.

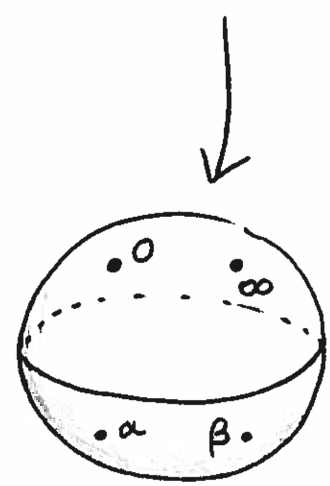
Geometric Picture:



What is the quotient here?

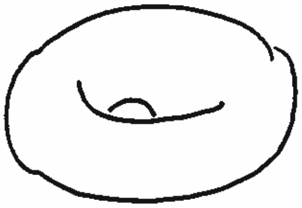
Each pt is equivalent to one on the back half.





It turns out this is exactly the picture of $C \longrightarrow \mathbb{P}^1_{\mathbb{C}}$!

Why is this plausible? Well, for example

 $= S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$

and $S^1 \leq \mathbb{C}^\times$ so this surface is also an abelian group. Moreover, π is locally a homeomorphism except at $(0,0), (0,\alpha), (0,\beta), \mathcal{O}$. where it looks like $\begin{array}{c} \mathbb{C} \longrightarrow \mathbb{C} \\ z \longmapsto z^2 \end{array}$. This is a "2-fold cover of $\mathbb{P}^1_{\mathbb{C}}$ branched at 4 points."

It turns out this is the only such cover...