Lecture 15: Algebraically closed fields and the fundamental theorem of algebra.

Last time: K/F is a splitting field for $f(x) \in F[x]$ if

(a) f splits into linear factors in $K[x]$.

(b) K is minimal with respect to (a).

Thm: Any $f(x) \in F[x]$ has a splitting field K.
Moreover $[K:F] \leq (\deg f)!$

Addendum: Suppose K, K' are splitting fields for $f(x) \in F[x]$. Then $\exists$ an isomorphism $\psi: K \to K'$ with $\psi|_F = id_F$.

Pf: See §13.4, Thm 27 of text. Think $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$

Algebraically closed: Every poly in $K[x]$ has a root in K.
($\Rightarrow$ it splits completely.)

Ex: $\mathbb{C}$, $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ is algebraic over } \mathbb{Q}\}$

Know this is a field.

Non Ex: $\mathbb{Q}$, $\mathbb{R}$.

Fundamental Theorem of Algebra (Gauss 1816)

Every $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. [Talk about pf shortly.]

Cor: $\overline{\mathbb{Q}}$ is algebraically closed.

Pf of Cor: Suppose $f(x) \in \overline{\mathbb{Q}}[x]$. Let α in $\mathbb{C}$ be a root of f. Then $\overline{\mathbb{Q}}(\alpha)/\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}/\mathbb{Q}$ are algebraic $\Rightarrow \overline{\mathbb{Q}}(\alpha)/\mathbb{Q}$ is algebraic $\Rightarrow \alpha \in \overline{\mathbb{Q}}$.

Def: K is an algebraic closure of F if K/F is algebraic and K is algebraically closed.

Ex: $\overline{\mathbb{Q}}/\mathbb{Q}$, $\mathbb{C}/\mathbb{R}$ Non ex: $\mathbb{C}/\mathbb{Q}$, $\overline{\mathbb{Q}} \cap \mathbb{R}/\mathbb{Q}$

Thm: Any field F has an algebraic closure.

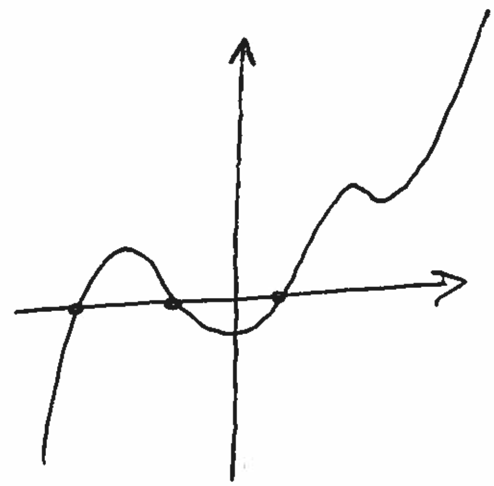
Pf: See Prop 31 of §13.4. Idea: keep adding in roots ad infinitum, e.g. using Zorn's Lemma.

Q: What is the alg. closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$? [Will discuss in 3 weeks]

<u>F.T.A.</u> Every nonconst $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$.

All proofs use some analysis/topology. Minimum is the following two consequences of the Intermediate Value Theorem:

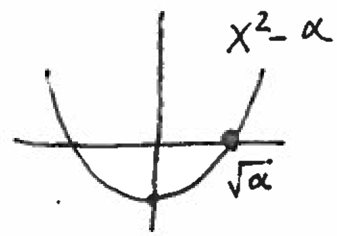
1. Every odd degree poly in $\mathbb{R}[x]$ has a root in $\mathbb{R}$. Reason:



$$f(x) = x^{\overset{\leftarrow \text{odd}}{d}} + a_{d-1}x^{d-1} + \dots + a_0$$

is pos for $x \gg 0$ and neg for $x \ll 0$.

2. Every pos α in $\mathbb{R}$ has a squareroot in $\mathbb{R}$.



Book proves FTA from these at the end of Sect 14.6 using Galois Theory.

[In some sense this proof misses the point...
There are many simpler proofs if you're willing to take a less algebraic point of view...]

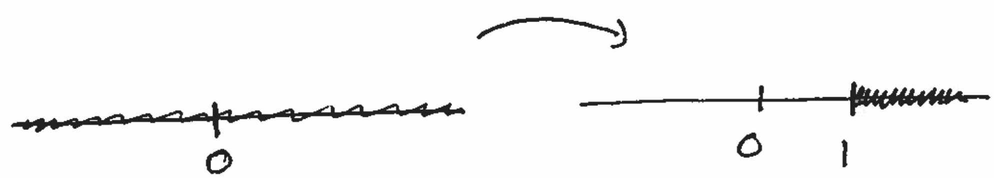
Today, just want to give an idea of what the issues are behind the F.T.A.

<u>Cor (FTA):</u> $f(z) \in \mathbb{C}[z]$ nonconstant. Then $f: \mathbb{C} \to \mathbb{C}$ is onto.

<u>Pf</u>: Given $w \in \mathbb{C}$, the poly $p(z) = f(z) - w$ has a root. $\square$

<u>Some worries about the FTA.</u>

1. Pleanty elts of $\mathbb{R}[x]$ don't have roots, e.g. x^2+1.



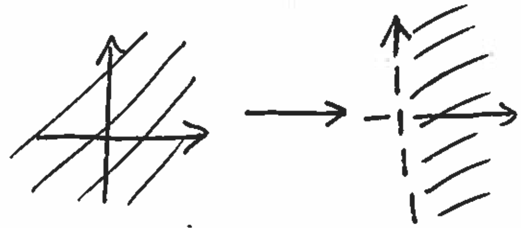
2. $f: \mathbb{C} \to \mathbb{C}$ for $f \in \mathbb{C}[z]$ is a very nice fn $\mathbb{R}^2 \to \mathbb{R}^2$ (e.g. diffentiable) but many such aren't onto: $(x,y) \longmapsto (x^2+y^2, xy-1)$

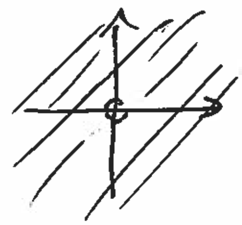
<u>So what's so special about polynomial maps $\mathbb{C} \to \mathbb{C}$?</u>

(A) For $f \in \mathbb{C}[z]$, its not too hard to show $f(\mathbb{C})$ is closed in $\mathbb{C}$, in contrast to:

$$\mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{and} \qquad \mathbb{C} \to \mathbb{C}$$

$$(x,y) \to (e^x, y) \qquad\qquad z \to e^z$$

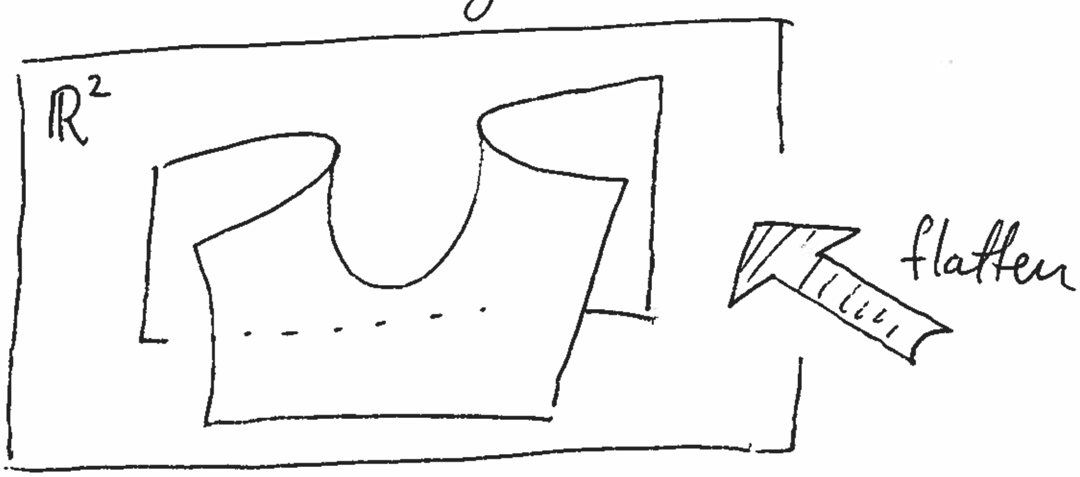


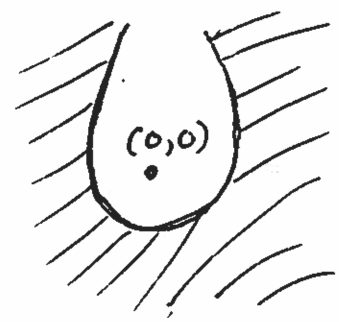


<u>Caution</u>: Also true for $f \in \mathbb{R}[x]$ as fns $\mathbb{R} \to \mathbb{R}$.

(B) For $f \in \mathbb{C}[z]$, the map $f: \mathbb{C} \to \mathbb{C}$ does not fold. [Go back to $x \mapsto x^2$ from $\mathbb{R} \to \mathbb{R}$.]

Can have similar thing for $\mathbb{R}^2 \to \mathbb{R}^2$





$f(\mathbb{C})$

In fact $f: \mathbb{C} \to \mathbb{C}$ "preserves angles".

Idea behind one proof of the F.T.A.

Suppose $0 \notin f(\mathbb{C})$.

By Ⓐ, there has to be a z_0 in $f(\mathbb{C})$ <u>closest</u> to 0. Now argue that f has to fold there for there not to be a closer point.