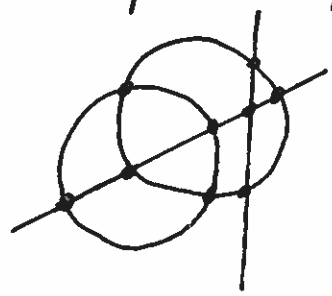
Lecture 13: Constructible Numbers

Rules: (A) Given two points, can draw the line joining them and the circle centered at one pt and passing through the other.

(B) Can find pts of intersection between drawn lines and circles.



[Gives midpts of segments, perpindicular bisectors, parallel lines.]

$\mathcal{C} = \left\{ z \in \mathbb{C} \;\middle|\; \begin{array}{l} z \text{ can be constructed from } \overset{0}{\bullet} \;\; \overset{1}{\bullet} \\ \text{by the above operations} \end{array} \right\}$

$\mathcal{C}$ is a field, closed under $|z|$, $\mathrm{Re}(z)$, $\mathrm{Im}(z)$, $\bar{z}$.

Thm A: If $z \in \mathcal{C}$, then $[\mathbb{Q}(z):\mathbb{Q}] = 2^n$. In particular, $\mathcal{C}/\mathbb{Q}$ is algebraic.

Cor: Can't construct a reg 7-gon.

Cor: Can't trisect angles

Cor: Can't square a circle.

[Today, will prove above theorem, as well as]

Thm B: $\mathcal{C}$ is the smallest subfield of $\mathbb{C}$ which is closed under taking square roots.

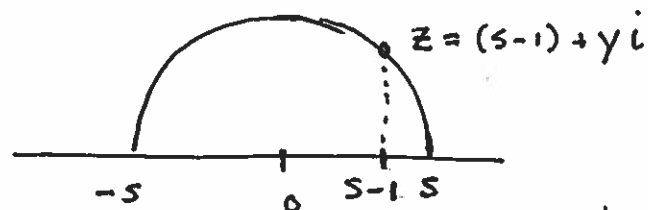
<u>Thm C</u>: $z \in \mathbb{C}$ is constructible iff there exist fields $\mathbb{Q} = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \mathbb{C}$ with $z \in K_n$ and each $[K_{k+1} : K_k] = 2$.

Note this gives Thm A as $2^n = [K_n : \mathbb{Q}] = [K_n : \mathbb{Q}(z)][\mathbb{Q}(z) : \mathbb{Q}]$.

<u>Lemma 1</u>: $\mathcal{C}$ is closed under $z \mapsto \sqrt{z}$

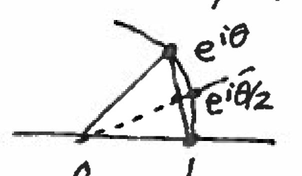
<u>Pf</u>: First, any $r \geq 1$ in $\mathcal{C} \cap \mathbb{R}$ has a square root as follows.

Set $s = \frac{r+1}{2}$ and consider



and note $x^2 + y^2 = s^2$

gives $y = \sqrt{s^2 - (s-1)^2} = \sqrt{2s-1} = \sqrt{r}$. As z in $\mathcal{C}$, have $\mathrm{Im}(z) = \sqrt{r} \in \mathcal{C}$ as well. Next any $0 \leq r < 1$ has $\sqrt{r} \in \mathcal{C}$ as $1/\sqrt{r} \in \mathcal{C}$. For the general $z = re^{i\theta} \in \mathcal{C}$, have $r = |z| \in \mathcal{C} \Rightarrow e^{i\theta} \in \mathcal{C}$. Since $\sqrt{z} = \sqrt{r}e^{i\theta/2}$, it remains to prove $e^{i\theta/2}$ in $\mathcal{C}$:



<u>Lemma 2</u>: Suppose $F \subseteq K \subseteq \mathbb{C}$. If $[K : F] = 2$, then $K = F(\sqrt{z})$ for some $z \in F$.

<u>Pf</u>. Pick $\alpha \in K \setminus F$. Then $K = F(\alpha)$ and $m_{\alpha, F}(x) = x^2 + bx + c$ for $b, c \in F$. By quadratic formula, have $K = F(\sqrt{b^2 - 4c})$ since $\alpha = (1 \pm \sqrt{b^2 - 4c})/2$.

[Explain why these lemmas give half of Thms B and C
Outline the moral.] ③

Set $P_1 = \{0, 1, i, -i\}$ and $P_n = \underbrace{\left\{ \begin{array}{l} \text{all } z \in \mathbb{C} \text{ constructible} \\ \text{in one step from pts in } P_{n-1} \end{array} \right\}}_{\text{finite set}}$

Define $F_n = \mathbb{Q}(P_n) \subseteq \mathbb{C}$.

Note $\bigcup F_n = \mathcal{C}$. By symmetry of P_1, have F_n closed under $z \mapsto \bar{z}$.

<u>Lemma 3</u>: For all $z \in P_n$, have $[F_n(z) : F_n] = 1$ or 2.

<u>Proof of Thm C</u>: ($\Leftarrow$) Clear from Lemmas 1 and 2

($\Rightarrow$) Since $\bigcup F_n = \mathcal{C}$, it suffices to show $[F_n : \mathbb{Q}]$ has such a tower of subfields.

Fix k, and number $z_1, z_2, \ldots, z_m$ in P_{k+1}. Consider

$$F_k \subseteq F_k(z_1) \subseteq F_k(z_1, z_2) \subseteq \cdots \subseteq F_k(z_1, \ldots, z_m) = F_{k+1}$$

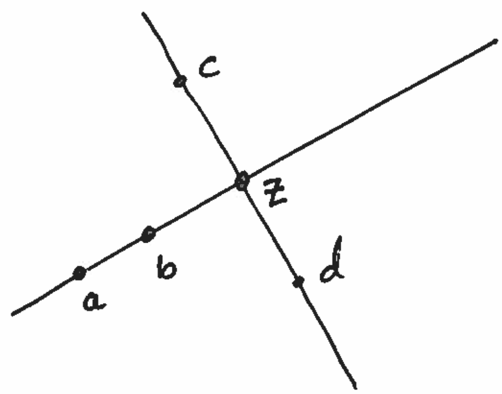
Since each z_i sat a poly of deg ≤ 2 in $F_k[x]$ by Lemma 3, have $[F_k(z_1, \ldots, z_i) : F_k(z_1, \ldots, z_{i-1})] = 1$ or 2

Removing duplicates gives the desired tower of subfields. $\square$

<u>Proof of Thm B</u>: Let K = smallest subfield of $\mathbb{C}$ that is closed under $\sqrt{\ }$. By Lemma 1, $K \subseteq \mathcal{C}$.

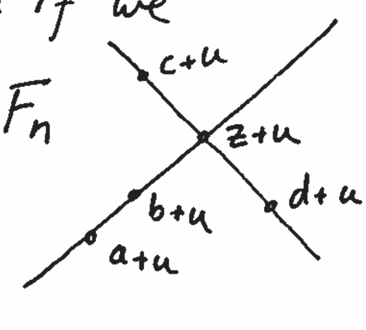
Conversely, any $z \in \mathcal{C}$ lives in a tower of subfields as given by Thm C. By Lemma 2, these are obtained by adding $\sqrt{\ }$'s, so $\mathcal{C} \subseteq K$. So $K = \mathcal{C}$. $\square$

<u>Proof of Lemma 3</u>:



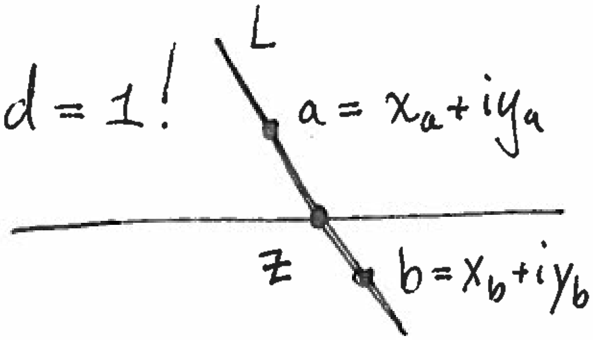
<u>Case 1</u>: $z \in P_n$ is the intersection of two lines defined by a, b, c, d in P_{n-1} as shown.

Set $K = F_n(z)$. Note that K is unchanged if we translate the whole picture by some $u \in F_n$



The same is true if we multiply the whole picture by $v \in F_n$.

Thus can assume that $c = 0$ and $d = 1$!

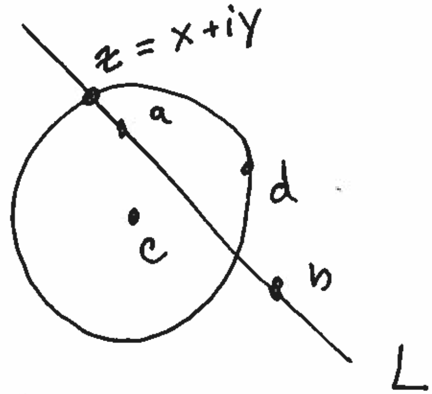


As F_n is closed under $w \mapsto \overline{w}$, it is also closed under $w \mapsto \mathrm{Re}(w) = \frac{1}{2}(w+\overline{w})$ and $w \mapsto \mathrm{Im}(w) = \frac{1}{2i}(w-\overline{w})$.

So $F_n \ni x_a, y_a, x_b, y_b$. The line L has eqn

$$(y_b - y_a)(x - x_a) = (x_b - x_a)(y - y_a)$$

and hence setting $y=0$ and solving for x shows that $z \in F_n$. So $K = F_n$.



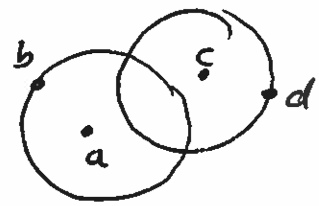
Case 2: $z \in P_n$ is the intersection of a line and a circle.

As before, assume $c=0$ and $d=1$. We seek the common solutions to $x^2+y^2=1$ and $y = mx+b$ for $m, b \in F_n$. Let (x, y) be the solution cor to z.

Note $(1+m^2)x^2 + 2mbx + b^2 - 1 = 0$, so $[F_n(x) : F_n] \leq 2$

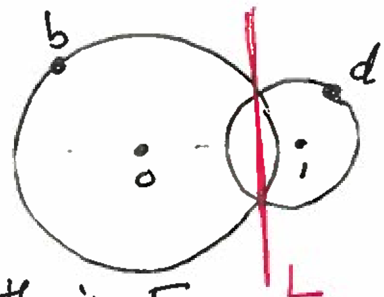
As $i \in F_n$, have $F_n(z) \subseteq F_n(x) = F_n(x,y)$ and so $[F_n(z) : F_n] \leq 2$.

Case z is the intersection of two circles



Can assume $a = 0$ and $c = 1$.

So consider



$$\left.\begin{aligned} x^2 + y^2 &= |b|^2 = b\bar{b} = R_1 \\ (x-1)^2 + y^2 &= |d-1|^2 = R_2 \end{aligned}\right\} \text{both in } F_n$$

Subtract to get $2x - 1 = R_1 - R_2 \Rightarrow x \in F_n$

and $F_n(z) = F_n(y) = F_n\left(\sqrt{R_1 - ((R_1 - R_2 + 1)/2)^2}\right)$.

So $[F_n(z) : F_n] \leq 2$. ▨

[Gauss-Wantzel 1830s] A regular n-gon is constructible if and only if $n = 2^k p_1 \cdots p_t$ where $k \geq 0$ and the p_i are distinct primes of the form $2^{2^n} + 1$.

Only 5 such Fermat primes are known: 3, 5, 17, 257, 65537.