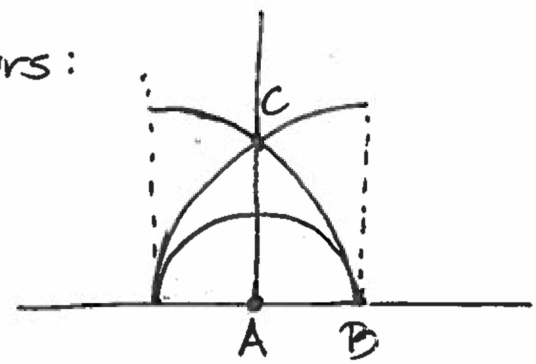
Lecture 12: Limitations of Straightedge and Compass.

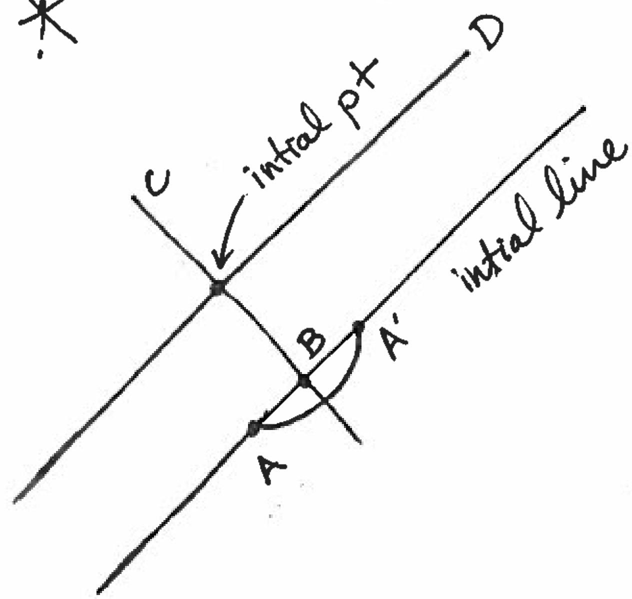
Given a ruler and compass, what can you construct?
[in the context of Euclidean plane geometry...]

① Divide a segment in half:

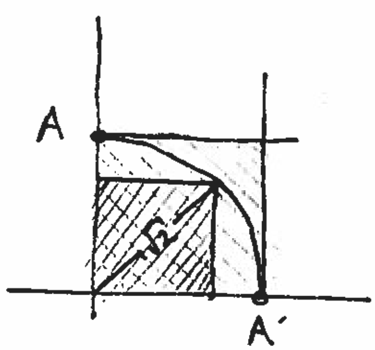
② Perpendicular bisectors:



③ Parallel lines:



④ Double the area of a square:



⑤ Make a regular 17 gon (or a 65537 gon).
[Shown by Gauss in 1796, first progress in 2000 years!)

Things you can't do:

1. Trisect an angle: 

2. Given a circle, construct a square with the same area.

<u>Rules</u>: [Goes back to the ancient Greeks.]

(A) Given two points, can draw

1. The line joining them.
2. The circle centered at one pt, passing throught the other.

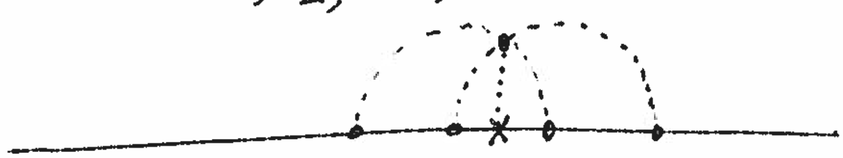
(B) Find the points of intersection of drawn lines and circles.

<u>Note</u>: Can't measure things.

<u>Constructable Numbers</u> Start with: 

$$C = \left\{ d \in \mathbb{R} \;\middle|\; \begin{array}{l} \text{From can construct two points} \\ a \text{ and } b \text{ with } \operatorname{dist}(a,b) = \pm d \end{array} \right\}$$

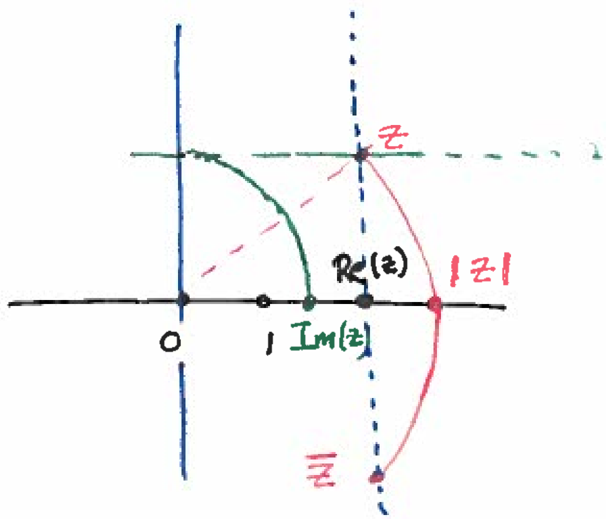
Some things in C: $\mathbb{Z}, \frac{1}{2}, \frac{\sqrt{3}}{2}, \ldots$



Choose coordinates to identify the plane with $\mathbb{C}$ so that the first pt is 0 and the other 1.

$\mathcal{C} = \{ z \in \mathbb{C} \mid$ the pt z is constructible from 0 and 1 $\}$

$\mathcal{C}_{\mathbb{R}} = \mathcal{C} \cap \mathbb{R}$



<u>Prop</u>: $\mathcal{C}$ is closed under

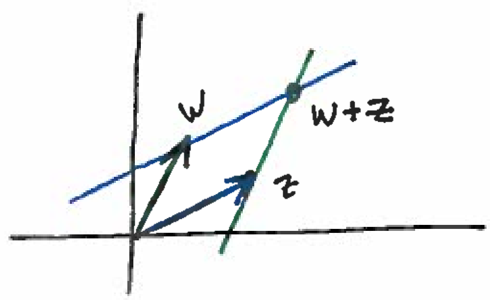
① $z \longmapsto |z|$

② $z \longmapsto \bar{z}$

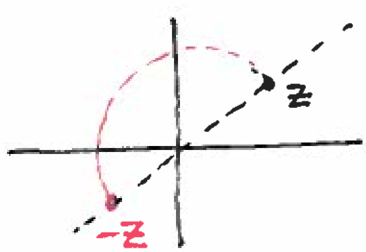
③ $z \longmapsto \mathrm{Re}(z)$

④ $z \longmapsto \mathrm{Im}(z)$

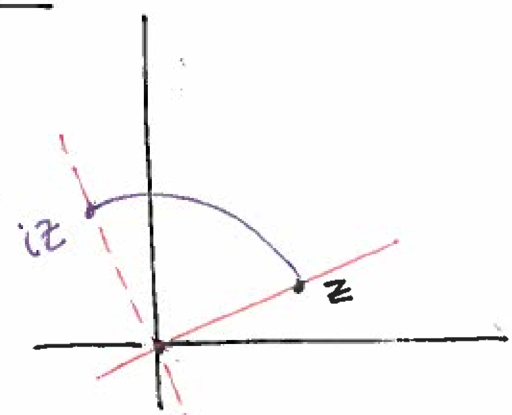
⑤ Addition



⑥ Subtraction



⑦ Mult by i



<u>Prop</u>: $z = x+iy \in \mathcal{C}$ iff x and y are in $\mathcal{C}_{\mathbb{R}}$

<u>Pf</u>: $(\Rightarrow)$ $z \in \mathcal{C} \Rightarrow \mathrm{Re}(z)$ and $\mathrm{Im}(z) \in \mathcal{C}_{\mathbb{R}}$

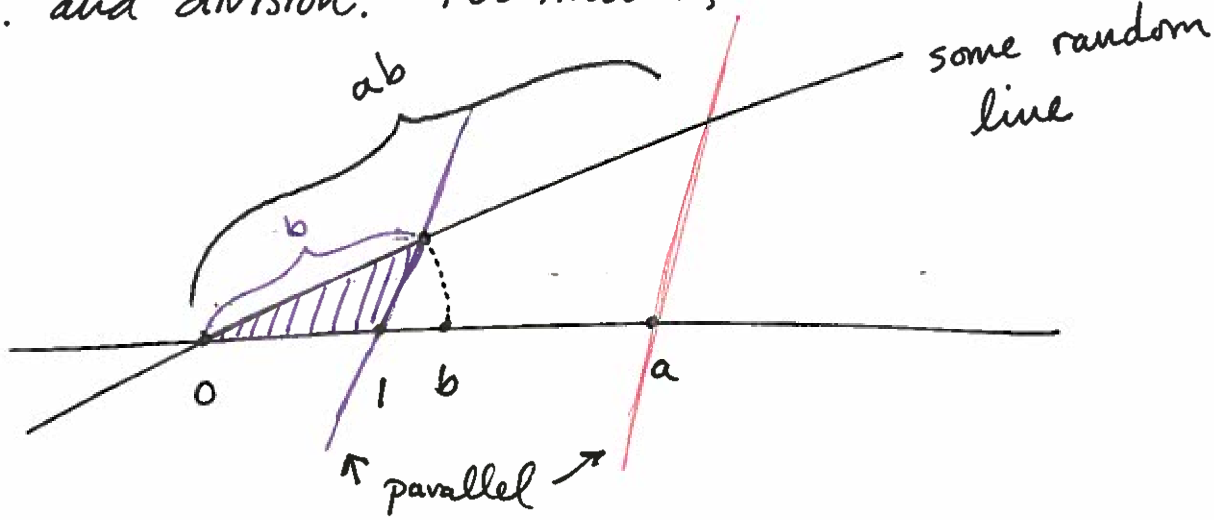
$(\Leftarrow)$ By ⑦ and ⑤ above.

Prop: $C = \mathcal{C}_{\mathbb{R}}$

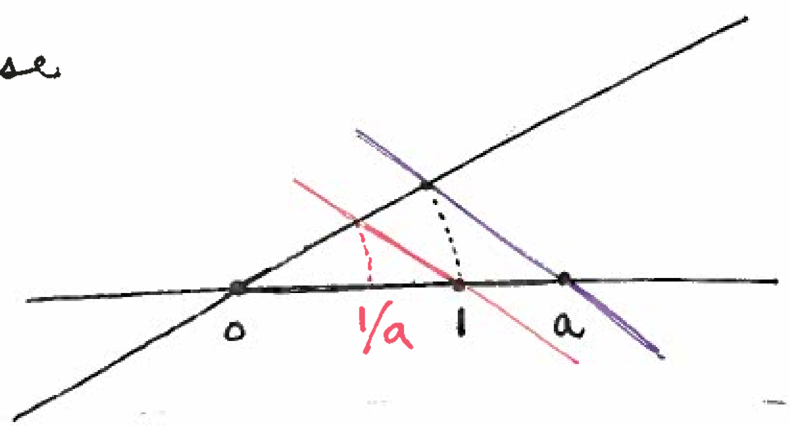
Pf: $\mathcal{C}_{\mathbb{R}} \subseteq C$ is clear. For $C \subseteq \mathcal{C}_{\mathbb{R}}$, note if $a, b \in \mathcal{C}$ then $\text{dist}(a,b) = |a-b|$. $\square$

Prop: $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}$ are fields.

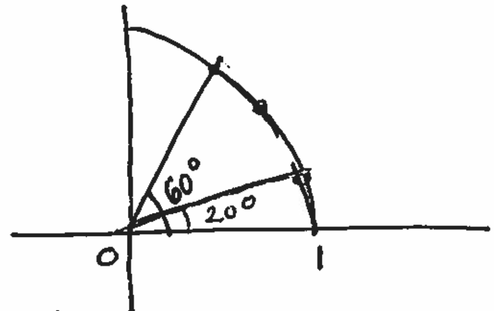
Pf: It suffices to prove $\mathcal{C}_{\mathbb{R}}$ is closed under mult. and division. For mult, do



To construct $1/a$ use



Thm: $z \in \mathcal{C}$. Then $[\mathbb{Q}(z):\mathbb{Q}] = 2^n$.

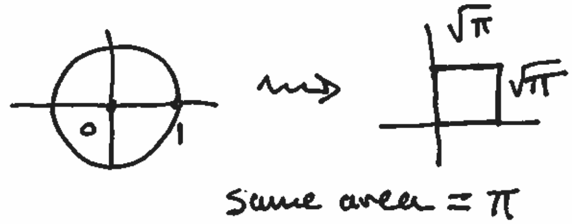


Cor: Can't trisect angles

Pf: If so, could construct
$z = e^{i\pi/9}$. Now z is a root of $x^6 - x^3 + 1$
which is irreducible in $\mathbb{F}_2[x]$ and hence $\mathbb{Q}[x]$.

So $[\mathbb{Q}(z):\mathbb{Q}] = 6$.

Cor: Can't square the circle.



Pf: For a unit circle, the sides
of the square would be $\sqrt{\pi}$. So $\sqrt{\pi} \in \mathcal{C}$
$\Rightarrow \pi \in \mathcal{C} \Rightarrow \pi$ algebraic, a contradiction.