1. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

   (a) Use Galois theory to prove that $\alpha = \sqrt{3} + \sqrt{7}$ is a primitive element for $K/\mathbb{Q}$, i.e. that $K = \mathbb{Q}(\alpha)$. **(6 points)**

   (b) Consider the $\mathbb{Q}$-linear transformation $T\colon K \to K$ where $T(\beta) = \alpha \cdot \beta$. Give the matrix A of T with respect to the $\mathbb{Q}$-basis $\{1, \sqrt{3}, \sqrt{7}, \sqrt{21}\}$ of K. **(2 points)**

   (c) Describe how you could use the marix A to find express α^{-1} as $a + b\sqrt{3} + c\sqrt{7} + d\sqrt{21}$, where $a, b, c, d \in \mathbb{Q}$. **(2 points)**

2. Let $\mathbb{Q} \subset K \subset \mathbb{C}$, where $K/\mathbb{Q}$ is a finite Galois extension. Let $\tau \in \text{Aut}(\mathbb{C})$ by complex conjugation. Prove or disprove: $\tau(K) = K$ and so τ gives an element of $\text{Gal}(K/\mathbb{Q})$.
(8 points)

3. Let R be a principal ideal domain.

(a) If α is an irreducible element of R, prove that the ideal $I = (\alpha)$ is maximal. **(4 points)**

(b) Prove that any proper ideal I of R is contained in a maximal ideal. **(6 points)**

(c) Does (a) remain true if R is just a UFD? Prove your answer. **(2 points)**

4. Consider the cyclotomic field $K = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/5}$. We know $K/\mathbb{Q}$ is Galois with group $G \cong (\mathbb{Z}/5\mathbb{Z})^{\times}$.

   (a) What is the minimal polynomial of ζ over $\mathbb{Q}$? **(2 points)**

   (b) How many subfields L of K are there with $[L : \mathbb{Q}] = 2$? **(2 points)**

   (c) Let $\sigma \in G$ send $\zeta \mapsto \zeta^2$. Find the corresponding fixed field $K_{\langle \sigma \rangle}$. **(4 points)**

   (d) Find the minimal polynomial of $\zeta^2 + \zeta^3$ over $\mathbb{Q}$. Your answer should not involve ζ. **(4 points)**

5. Let F be a field of characteristic 0. Let K be the splitting field of an irreducible *cubic* $f(x) \in F[x]$. Let $\alpha_1, \alpha_2, \alpha_3 \in K$ be the roots of f, and suppose that $G = \mathrm{Gal}(K/F)$ is all of S_3.

   (a) Show that $F = \mathbb{Q}$ and $f(x) = x^3 + x + 1$ is an example of this situation, i.e. that f is irreducible in $\mathbb{Q}[x]$ and $G = S_3$. **(4 points)**

   (b) Returning to the general case, for each j find the subgroup of G that corresponds to $F(\alpha_j)$. **(2 points)**

   (c) Prove that $F(\alpha_1) \cap F(\alpha_2) = F$. **(2 points)**

   (d) Prove that $\mathrm{Aut}(F(\alpha_1)/F)$ is trivial. **(4 points)**

   (e) Consider $\beta = \alpha_1\alpha_2^2 + \alpha_2\alpha_3^2 + \alpha_3\alpha_1^2$. Prove that $K \neq F(\beta)$. **(2 points)**

6. Consider the plane curve $X = \mathbf{V}(x^2 - y^2 - 1) \subset \mathbb{R}^2$.

(a) Prove that X is smooth, and draw a picture of it. **(4 points)**

(b) Let $\overline{X}$ be the corresponding curve in $\mathbb{P}^2_{\mathbb{R}}$. Find the defining equation for $\overline{X}$ in $\mathbb{R}[x, y, z]$, and find all the points in $\overline{X} - X$, i.e. all points at infinity. **(2 points)**

(c) Explain why your answers in (a) and (b) are consistent with the view that $\mathbb{P}^2_{\mathbb{R}}$ is $\mathbb{R}^2$ plus one point for each family of parallel lines in $\mathbb{R}^2$. **(2 points)**

(d) What is the topology of $\overline{X}$? What about if we replace with $\mathbb{R}$ with $\mathbb{C}$? You do not need to justify your answer, but should draw pictures. **(2 points)**

7. Let V be the plane curve $\mathbf{V}(x^2 - y^2 - 1) \subset \mathbb{C}^2$, which is irreducible. Let $K = \mathbb{C}(V)$ be the function field.

   (a) Consider the rational function on V given by

   $$f = \frac{x^2 - y - 1}{y - 1} \in K$$

   Prove that $\operatorname{dom}(f) = V$, even though the denominator vanishes at $(\sqrt{2}, 1) \in V$. **(4 points)**

   (b) Consider $h(x, y) = x$ in $\mathbb{C}[V]$ as a map $V \to \mathbb{C}$. Let $F = \mathbb{C}(\mathbb{C}) = \mathbb{C}(t)$, and consider $h^*\colon F \to K$ be the induced homomorphism of fields. As this is 1-1, identify F with its image under h^*. Describe the extension K/F as $F[u]/(p(u))$ for some *irreducible* polynomial $p(u) \in F[u]$. **(6 points)**

   (c) Is K/F Galois? If it is, describe how each element of $\mathrm{Gal}(K/F)$ acts on K. **(2 points)**

8. Throughout, let k be an algebraically closed field.

(a) Suppose $V_1, V_2 \subset k^n$ are affine varieties defined by $V_i = \mathbf{V}(I_i)$. Prove directly from the definitions that $V_1 \cup V_2 = \mathbf{V}(I_1 \cap I_2)$ **(4 points)**

(b) Let J_1 and J_2 be radical ideals in $k[x_1, \ldots, x_n]$. Prove that $I = J_1 \cap J_2$ is also a radical ideal, i.e. that $f^n \in I \Rightarrow f \in I$. **(2 points)**

(c) Show that $\mathbf{I}(V_1 \cup V_2) = \mathbf{I}(V_1) \cap \mathbf{I}(V_2)$. **(4 points)**