1. Let $K = \mathbb{Q}(\sqrt{3}, \sqrt{7})$.

(a) Use Galois theory to prove that $\alpha = \sqrt{3} + \sqrt{7}$ is a primitive element for $K/\mathbb{Q}$, i.e. that $K = \mathbb{Q}(\alpha)$. \hfill (6 points)

(b) Consider the $\mathbb{Q}$-linear transformation $T: K \to K$ where $T(\beta) = \alpha \cdot \beta$. Give the matrix $A$ of $T$ with respect to the $\mathbb{Q}$-basis $\{1, \sqrt{3}, \sqrt{7}, \sqrt{21}\}$ of $K$. \hfill (2 points)

(c) Describe how you could use the matrix $A$ to find express $\alpha^{-1}$ as $a + b\sqrt{3} + c\sqrt{7} + d\sqrt{21}$, where $a, b, c, d \in \mathbb{Q}$. \hfill (2 points)
2. Let $\mathbb{Q} \subset K \subset \mathbb{C}$, where $K/\mathbb{Q}$ is a finite Galois extension. Let $\tau \in \text{Aut}(\mathbb{C})$ by complex conjugation. Prove or disprove: $\tau(K) = K$ and so $\tau$ gives an element of $\text{Gal}(K/\mathbb{Q})$.

(8 points)
3. Let $R$ be a principal ideal domain.

(a) If $\alpha$ is an irreducible element of $R$, prove that the ideal $I = (\alpha)$ is maximal. \hspace{1cm} (4 \text{ points})

(b) Prove that any proper ideal $I$ of $R$ is contained in a maximal ideal. \hspace{1cm} (6 \text{ points})

(c) Does (a) remain true if $R$ is just a UFD? Prove your answer. \hspace{1cm} (2 \text{ points})
4. Consider the cyclotomic field $K = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i / 5}$. We know $K/\mathbb{Q}$ is Galois with group $G \cong (\mathbb{Z}/5\mathbb{Z})^\times$.

(a) What is the minimal polynomial of $\zeta$ over $\mathbb{Q}$? (2 points)

(b) How many subfields $L$ of $K$ are there with $[L : \mathbb{Q}] = 2$? (2 points)

(c) Let $\sigma \in G$ send $\zeta \mapsto \zeta^2$. Find the corresponding fixed field $K_{(\sigma)}$. (4 points)

(d) Find the minimal polynomial of $\zeta^2 + \zeta^3$ over $\mathbb{Q}$. Your answer should not involve $\zeta$. (4 points)
5. Let $F$ be a field of characteristic 0. Let $K$ be the splitting field of an irreducible cubic $f(x) \in F[x]$. Let $\alpha_1, \alpha_2, \alpha_3 \in K$ be the roots of $f$, and suppose that $G = \text{Gal}(K/F)$ is all of $S_3$.

(a) Show that $F = \mathbb{Q}$ and $f(x) = x^3 + x + 1$ is an example of this situation, i.e. that $f$ is irreducible in $\mathbb{Q}[x]$ and $G = S_3$. (4 points)

(b) Returning to the general case, for each $j$ find the subgroup of $G$ that corresponds to $F(\alpha_j)$. (2 points)

(c) Prove that $F(\alpha_1) \cap F(\alpha_2) = F$. (2 points)

(d) Prove that $\text{Aut}(F(\alpha_1)/F)$ is trivial. (4 points)

(e) Consider $\beta = \alpha_1\alpha_2^2 + \alpha_2\alpha_3^2 + \alpha_3\alpha_1^2$. Prove that $K \neq F(\beta)$. (2 points)
6. Consider the plane curve $X = V(x^2 - y^2 - 1) \subset \mathbb{R}^2$.

   (a) Prove that $X$ is smooth, and draw a picture of it.  \textbf{(4 points)}

   (b) Let $\overline{X}$ be the corresponding curve in $\mathbb{P}^2_{\mathbb{R}}$. Find the defining equation for $\overline{X}$ in $\mathbb{R}[x, y, z]$, and find all the points in $\overline{X} - X$, i.e. all points at infinity. \textbf{(2 points)}

   (c) Explain why your answers in (a) and (b) are consistent with the view that $\mathbb{P}^2_{\mathbb{R}}$ is $\mathbb{R}^2$ plus one point for each family of parallel lines in $\mathbb{R}^2$. \textbf{(2 points)}

   (d) What is the topology of $\overline{X}$? What about if we replace with $\mathbb{R}$ with $\mathbb{C}$? You do not need to justify your answer, but should draw pictures. \textbf{(2 points)}
7. Let $V$ be the plane curve $V(x^2 - y^2 - 1) \subset \mathbb{C}^2$, which is irreducible. Let $K = \mathbb{C}(V)$ be the function field.

(a) Consider the rational function on $V$ given by

$$f = \frac{x^2 - y - 1}{y - 1} \in K$$

Prove that $\text{dom}(f) = V$, even though the denominator vanishes at $(\sqrt{2}, 1) \in V$. (4 points)

(b) Consider $h(x, y) = x$ in $\mathbb{C}[V]$ as a map $V \to \mathbb{C}$. Let $F = \mathbb{C}(\mathbb{C}) = \mathbb{C}(t)$, and consider $h^*: F \to K$ be the induced homomorphism of fields. As this is 1-1, identify $F$ with its image under $h^*$. Describe the extension $K/F$ as $F[u]/(p(u))$ for some irreducible polynomial $p(u) \in F[u]$. (6 points)

(c) Is $K/F$ Galois? If it is, describe how each element of $\text{Gal}(K/F)$ acts on $K$. (2 points)
8. Throughout, let \( k \) be an algebraically closed field.

(a) Suppose \( V_1, V_2 \subset k^n \) are affine varieties defined by \( V_i = V(I_i) \). Prove directly from the definitions that \( V_1 \cup V_2 = V(I_1 \cap I_2) \) (4 points)

(b) Let \( J_1 \) and \( J_2 \) be radical ideals in \( k[x_1, \ldots, x_n] \). Prove that \( I = J_1 \cap J_2 \) is also a radical ideal, i.e. that \( f^n \in I \Rightarrow f \in I \). (2 points)

(c) Show that \( I(V_1 \cup V_2) = I(V_1) \cap I(V_2) \). (4 points)