1. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$ with $n \geq 3$ and let $G = \text{Gal}(K/\mathbb{Q})$.
   (a) Let $\tau \in G$ be the restriction of complex conjugation. Find the element that $\tau$ corresponds to under the isomorphism $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
   (b) Let $K^+ = K \cap \mathbb{R}$. Prove that $K^+ = \mathbb{Q}(\alpha)$ where $\alpha = \zeta_n + \zeta_n^{-1}$.

2. Find the Galois group of $x^4 - 7$ over $\mathbb{Q}$ explicitly as a permutation group on the roots
   \[ \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \left\{\sqrt[4]{7}, -\sqrt[4]{7}, i\sqrt[4]{7}, -i\sqrt[4]{7}\right\} \]
   Clarification: Your answer should both give the isomorphism type of $\text{Gal}(K/\mathbb{Q})$ and identify the explicit subgroup of $S_4 = \text{SymmetricGroup}(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ which is its image under the map $\text{Gal}(K/\mathbb{Q}) \to S_4$.

3. (a) For $I = (x^2 + 1) \subset \mathbb{R}[x]$, prove that $I$ is maximal and that $V(I) = \emptyset$.
    (b) Suppose that $I \subset \mathbb{R}[x]$ is maximal. Show that $V(I)$ is either empty or a single point.

4. Let $R = \mathbb{C}[x_1, \ldots, x_n]$.
   (a) If $f \in R$ is an irreducible polynomial prove that the variety $V(f) \subset \mathbb{C}^n$ is irreducible.
   (b) If $I \subset R$ is a proper ideal, prove that its radical is the intersection of all maximal ideals containing $I$.

5. Let $V$ be an algebraic variety in $\mathbb{k}^n$, and set $I = \mathcal{I}(V)$. Recall the coordinate ring of $V$ is
   \[ k[V] = \{ f: V \to k \mid f \text{ is the restriction of a polynomial in } k[x_1, \ldots, x_n] \} \]
   \[ = k[x_1, \ldots, x_n]/I \]
In particular, two elements of $k[V]$ are the same if they agree at every point of $V$, even if nominally they come from different polynomials.

The ring $k[V]$ contains $k$ as the subring of constant functions, coming from the polynomials with only a constant term. Thus it is a vector space over $k$. Prove that the following are equivalent.

(a) $V$ is a finite set of points in $k^n$.

(b) $k[V]$ is finite-dimensional as a $k$-vector space.

Hints: For (a) ⇒ (b) consider $k[V]$ as a subspace of the vector space of all functions $f : V \to k$. For (b) ⇒ (a), for each coordinate, try to show that the set \{\(a_i \mid (a_1, \ldots, a_n) \in V\)\} is finite.